UNIVERSITY OF CAMBRIDGE

MAMA/203, NST3AS/203, MAAS/203

MAT3

MATHEMATICAL TRIPOS Part III

Friday, 9 June, 2023 9:00 am to 11:00 am

PAPER 203

SCHRAMM-LOEWNER EVOLUTIONS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Give the definitions of a *compact* $\mathbb{H}$*-hull* A, its *mapping-out function* g_A and its *half-plane capacity* $\mathrm{hcap}(A)$.

(b) If $\mathbb{D}$ is the open unit disk, show that the density $p(z, e^{i\theta}), \theta \in [0, \pi]$, for the first exit distribution (with respect to Lebesgue measure) of a complex Brownian motion on $\mathbb{H} \cap \partial\mathbb{D}$ starting from $z \in \mathbb{H} \setminus \overline{\mathbb{D}}$ satisfies

$$p(z, e^{i\theta}) = \frac{2}{\pi} \frac{\mathrm{Im}(z)}{|z|^2} \sin(\theta)(1 + O(|z|^{-1})) \quad \text{as} \quad z \to \infty.$$

[You may use without proof that $f(z) = z + \frac{1}{z}$ *is a conformal transformation mapping* $\mathbb{H} \setminus \overline{\mathbb{D}}$ *onto* $\mathbb{H}$*. Also, you may use results from lectures provided you state them clearly.]*

(c) Show that if A is a compact $\mathbb{H}$-hull with $A \subseteq \overline{\mathbb{D}} \cap \mathbb{H}$ then

$$\mathrm{hcap}(A) = \frac{2}{\pi} \int_0^\pi \mathbb{E}_{e^{i\theta}} [\mathrm{Im}(B_\tau)] \sin(\theta) d\theta,$$

where τ is the first time that a complex Brownian motion B exits $\mathbb{H} \setminus A$ and $\mathbb{E}_z$ denotes the expectation with respect to the law under which B starts from z.

(d) (i) Consider the rectangle $A_r = [-r, r] \times (0, 1]$ in $\mathbb{H}$. Show that there exists constant $c > 0$ such that $\mathrm{hcap}(A_r) \leqslant cr$ for all $r \geqslant 1$.

(ii) Find a sequence of compact $\mathbb{H}$-hulls (A_n) such that $\mathrm{diam}(A_n) \to \infty$ but $\mathrm{hcap}(A_n) \to 0$ as $n \to \infty$.

2

(a) Suppose that γ is an SLE_κ in $\mathbb{H}$ from 0 to ∞ for $\kappa > 0$. State the conformal Markov property for the curve γ.

(b) Let X be a Bessel process with dimension $d > 0$, starting from $x > 0$. Show that if $d < 2$, then X_t hits 0 almost surely. Show also that if $d > 2$, then X_t does not hit 0 almost surely.

(c) Fix $\kappa \in (4, 8)$ and let γ be an SLE_κ in $\mathbb{H}$ from 0 to ∞. Let (g_t) be the family of conformal maps solving the chordal Loewner equation associated with γ. For each $x \in \mathbb{R} \setminus \{0\}$, let $V_t^x = g_t(x) - U_t$ for $t < \tau_x$, where $\tau_x = \inf\{t \geqslant 0 : V_t^x = 0\}$ and U_t is the driving function of γ.

(i) Fix $r > 1$ and for each $0 \leqslant t < \tau_1$, set $Z_t = \log\left(\frac{V_t^r - V_t^1}{V_t^1}\right)$ and consider the time-changed process $\tilde{Z}_t = Z_{\sigma(t)}$, where $\sigma(t) = \inf\left\{u \geqslant 0 : \int_0^u \frac{1}{(V_s^1)^2} ds = t\right\}$. Find the stochastic differential equation satisfied by $\tilde{Z}_t$. *[You may use Itô's formula. You may also assume without proof that $\sigma(\infty) = \tau_1$ almost surely.]*

(ii) Deduce that the following is true. Fix $x \in \mathbb{R} \setminus \{0\}$. Then γ does not hit x almost surely. *[Hint: You may use without proof that* $\sup_{t \geqslant 0}(B_t + at)$ *is finite almost surely, where B_t is a standard one-dimensional Brownian motion with $B_0 = 0$ and $a < 0$, and that $\tau_1 < \infty$ almost surely. Examine the behaviour of $\mathbb{P}[\tau_1 = \tau_{1+\epsilon}]$ as $\epsilon \to 0$. You may also use results from lectures provided you state them clearly.]*

3

(a) Specify (without proof) for which range of κ values SLE_κ is simple, self-intersecting but not space-filling, and is space-filling.

(b) Prove that SLE_κ is simple for the range of values that you have stated in part (a) that it is simple. *[You may use properties of Bessel processes proved in class provided you state them clearly.]*

(c) Let X be a Bessel process with dimension $d > 0$, starting from $x > 0$. Fix $r > 0$ and set $Y_t = rX_{t/r^2}$ for $t \geqslant 0$. Show that (Y_t) has the law of a Bessel process of dimension d, starting from rx.

(d) Fix $\kappa \in (4, 8)$ and let $(K_t)_{t \geqslant 0}$ be the family of compact $\mathbb{H}$-hulls corresponding to an SLE_κ process γ in $\mathbb{H}$ from 0 to ∞.

(i) Fix $\epsilon \in (0,1)$. Show that there is a finite and positive constant $r_0 > 0$ such that for all $t \geqslant 0$

$$\mathbb{P}[\{z \in \mathbb{H} : |z| < r_0\sqrt{t}\} \subseteq K_t] \geqslant 1 - \epsilon.$$

[Hint: You may use without proof that almost surely there is a (random) $r > 0$ such that $\{z \in \mathbb{H} : |z| < r\} \subseteq K_1$.]

(ii) Show that γ is almost surely transient, i.e., $\mathbb{P}[\liminf_{t\to\infty} |\gamma(t)| = \infty] = 1$. *[Hint: You may assume without proof that almost surely $\gamma((t,\infty)) \subseteq \overline{\mathbb{H} \setminus K_t}$ for each $t \geqslant 0$.]*

4 We assume throughout that $D \subseteq \mathbb{C}$ is a simply connected domain distinct from $\mathbb{C}$ and $\emptyset$.

(a) Give the definitions of:

(i) The space $C_0^\infty(D)$.

(ii) The space $H_0^1(D)$.

(iii) The Gaussian free field (GFF) h on D.

(b) State and prove the Markov property of the GFF h on D.

(c) (i) Give the definition of the Green's function $G_D(\cdot, \cdot)$ on D.

(ii) Explain how the L^2 inner product (h, ϕ) is defined for $\phi \in C_0^\infty(D)$ and a GFF h on D, and show that it is a mean-zero normal random variable with variance $\int_D \int_D \phi(x) G_D(x, y) \phi(y) dx dy$. *[You may assume without proof that $-2\pi\Delta^{-1}\phi(x) = \int_D G_D(x, y)\phi(y)dy$.]*

(d) Let ρ be a non-negative Borel measure with compact support in D so that $\int_D \int_D G_D(x, y)\rho(dx)\rho(dy) < \infty$. Show that (h, ρ) is a well-defined random variable in L^2 where h is a GFF on D. Show further that (h, ρ) is a Gaussian random variable with zero mean and variance $\int_D \int_D G_D(x, y)\rho(dx)\rho(dy)$.

END OF PAPER