MAT3

MATHEMATICAL TRIPOS Part III

Tuesday, 13 June, 2023 9:00 am to 12:00 pm

PAPER 202

STOCHASTIC CALCULUS AND APPLICATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **SIX** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let B be a standard Brownian motion with $B_0 = 0$. For each $\epsilon > 0$, let $f_\epsilon(x) = \sqrt{\epsilon^2 + x^2}$.

(a) Give the definition of a *semimartingale* and *ucp convergence.*

(b) Use Itô's formula to show that

$$f_\epsilon(B_t) = \epsilon + \int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s + \frac{1}{2}\int_0^t \frac{\epsilon^2}{(\epsilon^2 + B_s^2)^{3/2}} ds.$$

(c) Prove that $f_\epsilon(B_t)$ converges ucp to $|B_t|$ as $\epsilon \to 0$.

(d) Prove that

$$\int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s \to \int_0^t \operatorname{sign}(B_s) dB_s \quad \text{ucp} \quad \text{as} \quad \epsilon \to 0.$$

(e) Deduce that

$$\frac{1}{2}\int_0^t \frac{\epsilon^2}{(\epsilon^2 + B_s^2)^{3/2}} ds$$

converges ucp as $\epsilon \to 0$.

(f) Show that $|B_t|$ is a semimartingale.

2 Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions.

(a) Give the definition of the *previsible σ-algebra $\mathcal{P}$.*

(b) Give the definition of a *simple process.* Let ν be a finite measure on $(\Omega, \mathcal{F})$ and prove that the collection of simple processes is dense in $L^2(\mathcal{P}, \nu)$.

(c) Prove that $\mathcal{P}$ is the smallest σ-algebra which makes all left-continuous adapted processes measurable.

(d) Suppose that X is a previsible process and f is a Borel function. Prove that $f(X)$ is a previsible process.

(e) Prove that every deterministic càdlàg function is previsible.

(f) Give an example of a càdlàg process which is not previsible.

3

(a) Define the *stochastic exponential* $\mathcal{E}(M)$ of $M \in \mathcal{M}_{c,loc}$ with $M_0 = 0$, show that $\mathcal{E}(M) \in \mathcal{M}_{c,loc}$, and show that $\mathcal{E}(M)$ is also a supermartingale.

(b) Suppose that B is a standard Brownian motion with $B_0 = 0$. For each $a > 0$, let $T_a = \inf\{t \geqslant 0 : B_t + t = a\}$. Show that $\mathbb{E}[\exp(T_a/2)] = \exp(a)$.

[Hint: use Girsanov's theorem.]

(c) For each $b < 0$, let $S_b = \inf\{t \geqslant 0 : B_t - t = b\}$. Deduce from part (b) that $\mathbb{E}[\exp(S_b/2)] = \exp(-b)$ and hence show that if R is any stopping time for the filtration generated by B then $\mathbb{E}[\exp(B_{R\wedge S_b} - R \wedge S_b/2)] = 1$.

[Hint: show that $\mathcal{E}(B)_{t\wedge S_b}$ is a UI martingale.]

(d) Show that if $M \in \mathcal{M}_{c,loc}$ with $M_0 = 0$, $T > 0$, and $\mathbb{E}[\exp([M]_T/2)] < \infty$ then $\mathcal{E}(M^T)$ is a martingale.

[Hint: consider $\mathbb{E}[\mathbf{1}_{S_b \leqslant [M]_t} \exp(b + S_b/2)] + \mathbb{E}[\mathbf{1}_{[M]_t < S_b} \exp(M_t - [M]_t/2)]$ where B is defined by $B_s = M_{\tau_s}$ and $\tau_s = \inf\{t \geqslant 0 : [M]_t > s\}$.]

4 Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space satisfying the usual conditions.

(a) Suppose that $X = X_0 + M + A$ is a semimartingale where $[M]$ is strictly increasing with $\mathbb{P}[[M]_\infty = \infty] = 1$. For each $s \geqslant 0$ let $\tau_s = \inf\{t \geqslant 0 : [M]_t = s\}$, $\widetilde{\mathcal{F}}_s = \mathcal{F}_{\tau_s}$, and $\widetilde{X}_s = X_{\tau_s}$. Show that $\widetilde{X}$ is a continuous semimartingale adapted to $(\widetilde{\mathcal{F}}_s)$.

[You may use results proved in the course provided you state them clearly.]

(b) Use Itô's formula to find the semimartingale decomposition of $Z_t = \exp(B_t + at)$ where B is a standard Brownian motion and $a \in \mathbb{R}$.

(c) Recall that a d-dimensional Bessel process is given by the maximal local solution $(X, \mathcal{T})$ in $U = (0, \infty)$ of the SDE

$$dX_t = \frac{d-1}{2}\frac{1}{X_t}dt + dB_t.$$

Show that Z from part (b) is, up to a time change you should specify, given by a d-dimensional Bessel process. Derive the relationship between d and a.

(d) Using the previous part, show that $\mathbb{P}[\mathcal{T} < \infty] = 1$ if and only if $d < 2$. *[You may use without proof that* $\lim_{t\to\infty}(B_t + at) = -\infty$ *if and only if* $a < 0$.*]*

(e) Show that if $(X, \mathcal{T})$ is a d-dimensional Bessel process and $\alpha > 0$ then $(X^\alpha, \mathcal{T})$ is a d'-dimensional process, up to a time change. Derive the relationship between d and d'.

5 Let $v \in C^1((0,1)) \cap L^1((0,1))$. Consider the SDE

$$dX_t = \frac{1}{2}v(X_t)dt + dB_t \quad \text{with} \quad X_0 \in (0,1). \tag{1}$$

(a) Show that (1) has a pathwise unique maximal local solution $(X, \mathcal{T})$ in $U = (0,1)$ where $\mathcal{T} = \inf\{t \geqslant 0 : X_t \in \{0,1\}\}$. *[You may use results proved in the course provided you state them clearly.]*

(b) Define $V: (0,1) \to \mathbb{R}$ and $\psi: (0,1) \to \mathbb{R}$ by

$$V(t) = \int_0^t v(s)ds \quad \text{and} \quad \psi(x) = \int_0^x e^{-V(u)}du.$$

Show that $\psi(X_t)$ is a continuous local martingale.

(c) Show that $\mathbb{P}[\mathcal{T} < \infty, \ X_{\mathcal{T}} = 0] > 0$.

(d) Prove that for each $t > 0$ the law of $X_t \mathbf{1}_{\{t<\mathcal{T}\}}$ is absolutely continuous with respect to Lebesgue measure on $(0,1)$.

(e) Let $p(t,x,\cdot)$ be the Radon-Nikodym derivative of the law of law of $X_t \mathbf{1}_{\{t<\mathcal{T}\}}$ with respect to Lebesgue measure on $(0,1)$ when $X_0 = x$. Show that p satisfies the equation

$$\partial_t p(t,x,y) = \frac{1}{2}\partial_x^2 p(t,x,y) + \frac{1}{2}v(x)\partial_x p(t,x,y).$$

[You may assume without proof that $p(\cdot,\cdot,y) \in C^{1,2}(\mathbb{R}_+ \times (0,1))$ for each fixed $y \in (0,1)$.]

6

(a) Suppose that $D \subseteq \mathbb{R}^2$ is a bounded domain. Suppose that $u \in C(\overline{D}) \cap C^2(D)$ satisfies

$$\Delta u = 0 \quad \text{on} \quad D \quad \text{and} \quad u = f \quad \text{on} \quad \partial D$$

where $f \in C(\partial D)$. Let B be a standard Brownian motion and let $\tau = \inf\{t \geqslant 0 : B_t \notin D\}$. Show that

$$u(x) = \mathbb{E}_x[f(B_\tau)] \quad \text{for each} \quad x \in D.$$

(b) Prove or disprove that the assertion of part (a) holds if we replace the assumption that D is bounded with $\mathbb{P}[\tau < \infty] = 1$.

(c) Suppose that u is a bounded Borel function on $\mathbb{R}^2$ such that the following is true. For a standard Brownian motion B and every $x \in \mathbb{R}^2$ we have that $u(B_t + x) \in \mathcal{M}_{c,loc}$. Prove that u is constant.

END OF PAPER