UNIVERSITY OF CAMBRIDGE

MAMA/201, NST3AS/201, MAAS/201

MAT3

MATHEMATICAL TRIPOS Part III

Monday, 5 June, 2023 1:30 pm to 4:30 pm

PAPER 201

ADVANCED PROBABILITY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **SIX** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let Z be a random variable with $|Z| \leqslant 1$. Let $(\mathcal{F}_n)$ be a filtration and let $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n)$.

(a) Show that $(\mathbb{E}[Z \mid \mathcal{F}_n])$ is a martingale with respect to $(\mathcal{F}_n)$.

(b) Prove that $\mathbb{E}[Z \mid \mathcal{F}_n] \to \mathbb{E}[Z \mid \mathcal{F}_\infty]$ as $n \to \infty$ almost surely and in $\mathcal{L}^1$.

[You may assume the almost sure martingale convergence theorem.]

(c) Let $(X_n)_n$ be a sequence of random variables with $|X_n| \leqslant 1$ for all n and $X_n \to X$ as $n \to \infty$ almost surely and in $\mathcal{L}^1$. Prove that

$$\mathbb{E}[X_n \mid \mathcal{F}_n] \to \mathbb{E}[X \mid \mathcal{F}_\infty] \text{ as } n \to \infty \text{ almost surely and in } \mathcal{L}^1.$$

[Hint: Define $Z_n = \sup_{m \geqslant n} |X_m - X|$ and show first that $Z_n \to 0$ almost surely and in $\mathcal{L}^1$.]

2

State the almost sure martingale convergence theorem. Suppose that $(X_n)_{n \in \mathbb{N}}$ is a nonnegative bounded supermartingale. Show that

$$\mathbb{E}\Big[\lim_{n\to\infty} X_n\Big] = \lim_{n\to\infty} \mathbb{E}[X_n]\,.$$

Suppose that X is a discrete time simple symmetric random walk in $\mathbb{Z}^3$, i.e. $X_n = \sum_{i=1}^n \xi_i$, where (ξ_i) is an i.i.d. sequence of random variables with distribution

$$\mathbb{P}(\xi_1 = e_i) = \mathbb{P}(\xi_1 = -e_i) = \frac{1}{6} \quad \forall\, i \in \{1,2,3\},$$

where e_i is the i-th standard basis vector. Let $(\mathcal{F}_n)$ be its natural filtration.

(a) For every $x \in \mathbb{Z}^3$ define

$$v(x) = \mathbb{P}_x(\exists\, n \geqslant 0 : X_n = 0)\,.$$

Prove that $(v(X_n))$ is a supermartingale with respect to the filtration $(\mathcal{F}_n)$.

(b) Prove that $v(X_n) \to 0$ as $n \to \infty$ almost surely.

[You may assume that $\mathbb{P}_0(X_n \text{ visits } 0 \text{ infinitely often}) = 0$. You may use theorems from the course as long as they are stated clearly.]

3

Let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. random variables distributed according to the Poisson distribution with parameter 1. Set for every $n \geqslant 1$

$$S_n = \sum_{i=1}^{n} X_i \quad \text{and} \quad Y_n = \frac{S_n - n}{\sqrt{n}}.$$

(a) Prove that S_n has the Poisson distribution with parameter n. Writing $x^- = \max(-x, 0)$ prove that for all $a > 0$

$$\mathbb{P}\big(Y_n^- \geqslant a\big) \leqslant \frac{1}{a^2}.$$

(b) Let Y be a standard normal random variable. Show that (Y_n^-) converges weakly to Y^-.

(c) Show that $\mathbb{E}[Y_n^-] \to \mathbb{E}[Y^-]$ as $n \to \infty$.

[You may use the Cauchy Schwartz inequality $\mathbb{E}[XY] \leqslant \sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}$ for random variables X, Y without proof.]

(d) Deduce Stirling's formula, i.e. that as $n \to \infty$

$$n! \sim \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n},$$

where we write $a_n \sim b_n$ to mean $a_n/b_n \to 1$ as $n \to \infty$.

4

Let (B_t) be a standard Brownian motion in 1 dimension and let $(\mathcal{F}_t)$ be its natural filtration. For $\lambda \in \mathbb{R}$ and $t \geqslant 0$ we define

$$M_\lambda(t) = \exp(\lambda B_t - \lambda^2 t/2).$$

(a) Prove that for every $\lambda \in \mathbb{R}$ the process $(M_\lambda(t))_{t \geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)$.

(b) Prove carefully that for all $n \in \mathbb{N}$ the process $(d^n M_\lambda(t)/d\lambda^n)_{t \geqslant 0}$ is also a martingale with respect to $(\mathcal{F}_t)_t$.

(c) For every $x \in \mathbb{R}$ denote by τ_x the first hitting time of x, i.e.

$$\tau_x = \inf\{t \geqslant 0 : B_t = x\}.$$

Let $a, b > 0$. By differentiating $M_\lambda(t)$ in λ three times and then setting $\lambda = 0$ or otherwise prove that

$$\mathbb{E}[\tau_b \mid \tau_b < \tau_{-a}] = \frac{b^2 + 2ab}{3}.$$

You should justify carefully each application of the optional stopping theorem.

5

Let B be a standard Brownian motion in one dimension. Set

$$M_t = \max_{0 \leqslant s \leqslant t} B_s, \quad \forall\, t \geqslant 0.$$

(a) State the reflection principle for Brownian motion and then deduce from it that for all $t \geqslant 0$

$$M_t \stackrel{d}{=} |B_t|.$$

(b) Show that almost surely there is a unique time $M^* \in [0, 1]$ such that

$$B_{M^*} = \max_{0 \leqslant s \leqslant 1} B_s.$$

(c) Let M^* be as in part (b). Prove that M^* has the arcsine distribution, i.e. that for all $s \in [0, 1]$

$$\mathbb{P}(M^* \leqslant s) = \frac{2}{\pi} \arcsin(\sqrt{s}).$$

6

Let $B = (B_t)_{t \geqslant 0}$ be a standard Brownian motion in $d \geqslant 1$ dimensions and let Δ be the Laplacian operator, i.e. if $f : \mathbb{R}^d \to \mathbb{R}$ is a function, then

$$\Delta f(x) = \sum_{i=1}^{d} \frac{\partial_i^2 f(x)}{\partial x_i^2}.$$

(a) Suppose $d = 1$. Prove that almost surely

$$\limsup_{t \to \infty} B_t = \infty \quad \text{and} \quad \liminf_{t \to \infty} B_t = -\infty.$$

Deduce that B visits 0 infinitely often almost surely.

[You may use Bluementhal's 0-1 law without proof provided you state it clearly.]

From now on we take $d \geqslant 2$.

(b) Let $\widetilde{B}$ be an independent Brownian motion started from $x \in \mathbb{R}^d$. Define a sequence of stopping times as follows: set $T_0 = 0$ and inductively for $i \leqslant d$ set

$$T_i = \inf\{s \geqslant T_{i-1} : B_s^i = \widetilde{B}_s^i\}.$$

In words, T_i is the first time the i-th coordinates become equal after the first time the $i-1$-st coordinates become equal. Prove that $T_d < \infty$ almost surely.

(c) Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is a bounded harmonic function, i.e. $\Delta f(x) = 0$ for all x. Prove that f is constant.

[Hint: Use a stopping time to construct on the same probability space a pair of Brownian motions starting from two different points that eventually agree.]

END OF PAPER