UNIVERSITY OF CAMBRIDGE

MAMA/160, NST3AS/160, MAAS/160

MAT3

MATHEMATICAL TRIPOS Part III

Monday, 5 June, 2023 1:30 pm to 4:30 pm

PAPER 160

REPRESENTATION THEORY OF SYMMETRIC GROUPS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

All representations on this exam are assumed to be finite-dimensional.
Unless otherwise stated, they are over the field $\mathbb{C}$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $n \in \mathbb{N}$ and $\lambda \vdash n$. Define $\mathcal{S}^\lambda := \langle e(t) \mid t \in \Delta^\lambda \rangle_{\mathbb{C}}$, that is, $\mathcal{S}^\lambda$ is the $\mathbb{C}$-vector space spanned by the polytabloids corresponding to λ-tableaux. For $(i,j) \in \mathcal{Y}(\lambda)$, define $c_{i,j}(\lambda) := j - i$.

(a) Prove that $\mathcal{S}^\lambda$ is a non-zero cyclic $\mathbb{C}S_n$–module.

(b) Fix any λ-tableau t.

(i) Suppose $\tau \in S_n$ is a transposition. Show that if $\tau \in R(t)C(t)$, then τ belongs to exactly one of $R(t)$ and $C(t)$.

(ii) Show that the number of transpositions in $R(t)$ minus the number of transpositions in $C(t)$ is $\sum_{(i,j)\in\mathcal{Y}(\lambda)} c_{i,j}(\lambda)$.

Define $C = \sum_{1 \leqslant i < j \leqslant n} (i\ j) \in \mathbb{C}S_n$. In (c), you may assume that $\mathcal{S}^\lambda$ is irreducible.

(c) (i) Briefly explain why $C \in Z(\mathbb{C}S_n)$ and hence C acts on $\mathcal{S}^\lambda$ by multiplication by some scalar $c(\lambda) \in \mathbb{C}$.

(ii) Prove that

$$c(\lambda) = \sum_{(i,j)\in\mathcal{Y}(\lambda)} c_{i,j}(\lambda).$$

(d) (i) Show that

$$\sum_{(i,j)\in\mathcal{Y}(\lambda)} h_{i,j}(\lambda) = \sum_{(i,j)\in\mathcal{Y}(\lambda)} d_{i,j}(\lambda),$$

where $d_{i,j}(\lambda) = i + j - 1$. Hence, or otherwise, show that

$$\sum_{i\in\mathbb{N}} \lambda_i^2 = \sum_{(i,j)\in\mathcal{Y}(\lambda)} \big(h_{i,j}(\lambda) + c_{i,j}(\lambda)\big).$$

(ii) Deduce that

$$\sum_{i\in\mathbb{N}} \big(\lambda_i^2 + (\lambda')_i^2\big) = 2 \sum_{(i,j)\in\mathcal{Y}(\lambda)} h_{i,j}(\lambda),$$

and

$$\sum_{\mu\vdash n} \sum_{i\in\mathbb{N}} \mu_i^2 = \sum_{\mu\vdash n} \sum_{(i,j)\in\mathcal{Y}(\mu)} h_{i,j}(\mu).$$

(iii) Show that

$$\sum_{(i,j)\in\mathcal{Y}(\lambda)} h_{i,j}(\lambda)^2 = n^2 + \sum_{(i,j)\in\mathcal{Y}(\lambda)} c_{i,j}(\lambda)^2.$$

2 For a partition α, let χ^α denote the character of the irreducible α-Specht module over $\mathbb{C}$. In the usual notation from lectures, $\psi^\lambda = \sum_{\pi \in S_{\mathbb{N}}} \operatorname{sgn}(\pi) \cdot \xi^{\lambda - \mathrm{id} + \pi}$ for integer compositions λ, and id is the sequence $(1, 2, 3, \dots)$. Throughout, let $n \in \mathbb{N}$.

(a) (i) State the Murnaghan–Nakayama Rule.

(ii) Let λ be an integer composition of n. Let $i \in \mathbb{N}$ and define μ to be the integer composition satisfying $\mu - \mathrm{id} = (i \;\; i+1) \circ (\lambda - \mathrm{id})$. Show that $\psi^\mu = -\psi^\lambda$.

(iii) Let $k \in \{1, 2, \dots, n\}$. Let $\alpha = (\alpha_1, \alpha_2, \dots) \vdash n$. For each $i \in \mathbb{N}$ and $m \in \mathbb{N}_0$, define $\beta_{i,m}$ to be the sequence

$$\beta_{i,m} := (\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1} - 1, \dots, \alpha_{i+m} - 1, \alpha_i - k + m, \alpha_{i+m+1}, \dots).$$

Fix $i \in \mathbb{N}$. Prove that $\psi^{\beta_{i,0}} = 0$ if $\beta_{i,m}$ is not a partition for any $m \in \mathbb{N}_0$.

In parts (b)–(d) below, you may use general results from the course without proof, provided they are stated clearly.

(b) Let $\chi \in \operatorname{Irr}(S_n)$ and $g \in S_n$. If $\chi(g) \neq 0$, show that the order of g divides $\frac{|S_n|}{\chi(1)}$.

(c) Show that for every partition $\lambda \vdash n$ there exists $g \in S_n$ such that $\chi^\lambda(g) \in \{\pm 1\}$.

(d) Define F to be the virtual character

$$F := \chi^{(n)} - \chi^{(n-1,1)} + \chi^{(n-2,2)} - \cdots + (-1)^m \cdot \chi^{(n-m,m)}$$

of S_n, where $m = \lfloor \frac{n}{2} \rfloor$.

(i) Show that there exists $g \in S_n$ such that $F(g) \neq 0$ and g contains a cycle of length k in its disjoint cycle decomposition satisfying $k \equiv n \pmod 2$.

(ii) If n is even, prove that $F(g) = 0$ for all $g \in S_n$ whose disjoint cycle decomposition contains a cycle of odd length.

(iii) Give, with justification, an example of each of the following:

* an odd n and $g \in S_n$ such that g contains a cycle of even length in its disjoint cycle decomposition and $F(g) \neq 0$;
* an odd n and $g \in S_n$ such that g contains a cycle of even length in its disjoint cycle decomposition and $F(g) = 0$.

3 Given a partition λ and $e \in \mathbb{N}$, let $Q_e(\lambda) = (\lambda^{(0)}, \ldots, \lambda^{(e-1)})$, $C_e(\lambda)$ and $\mathbf{w}_e(\lambda)$ denote the e-quotient, e-core and e-weight of λ, respectively.

(a) (i) Let $\lambda \vdash n$ and $e \in \mathbb{N}$. Prove that there is a bijection

$$f : \{H_{i,j}(\lambda) \text{ s.t. } e \text{ divides } h_{i,j}(\lambda)\} \to \{\text{hooks in } Q_e(\lambda)\}$$

such that if $H = H_{i,j}(\lambda)$ with $e \mid h_{i,j}(\lambda)$, then $|H| = e \cdot |f(H)|$, and moreover that $Q_e(\lambda \setminus H) = Q_e(\lambda) \setminus f(H)$.

[You may assume the following facts without proof: if $\mathbf{X} = \{h_1, \ldots, h_m\}$ is a β-set for λ, then $h \in \mathcal{H}_i(\lambda)$ if and only if $0 \leqslant h_i - h \notin \mathbf{X}$. Also, $(\mathbf{X} \setminus \{h_i\}) \sqcup \{h_i - h_{i,j}(\lambda)\}$ is a β-set for $\lambda \setminus H_{i,j}(\lambda)$.]

(ii) Let $\lambda = (6, 5, 3)$ and $e = 3$. For all $(i, j) \in \mathcal{Y}(\lambda)$ such that $e \mid h_{i,j}(\lambda)$ and f as defined in (i), calculate $f(H_{i,j}(\lambda))$ in the form $H_{i',j'}(\lambda^{(s)})$.

In parts (b) and (c) below, you may use general results from the course without proof, provided they are stated clearly.

(b) Is each of the following statements (i) – (iv) true for all partitions λ and all natural numbers e and f? Justify your answers.

[In parts (iii) and (iv), if $P = (P_1, \ldots, P_t)$ is a sequence of t partitions, then $C_e(P)$ denotes the sequence $(C_e(P_1), \ldots, C_e(P_t))$, while $Q_e(P)$ denotes the concatenation of the sequences $Q_e(P_1), \ldots, Q_e(P_t)$ (so that $Q_e(P)$ is a sequence of te partitions).]

(i) If $\lambda = C_e(\lambda)$, then $\lambda = C_{ef}(\lambda)$.

(ii) $C_e(C_f(\lambda)) = C_f(C_e(\lambda))$.

(iii) $Q_e(Q_f(\lambda))$ is a permutation of $Q_{ef}(\lambda)$.

(iv) $C_e(Q_f(\lambda)) = Q_f(C_{ef}(\lambda))$.

(c) Let $\lambda \vdash n$ and $e \in \mathbb{N}$. Suppose $w \in \mathbb{N}_0$ satisfies $w \geqslant \mathbf{w}_e(\lambda)$ and $we \leqslant n$. Suppose $\rho \in S_n$ is a product of w disjoint e-cycles and $\gamma \in S_{n-we}$ is disjoint from ρ. Prove that

$$\chi^\lambda(\rho\gamma) = \begin{cases} \varepsilon \cdot \binom{w}{|\lambda^{(0)}|, \ldots, |\lambda^{(e-1)}|} \cdot \chi^{C_e(\lambda)}(\gamma) \cdot \prod_{i=0}^{e-1} \chi^{\lambda^{(i)}}(1) & \text{if } w = \mathbf{w}_e(\lambda), \\ 0 & \text{if } w > \mathbf{w}_e(\lambda), \end{cases}$$

where $\varepsilon \in \{\pm 1\}$. Here $\binom{w}{a,b,\ldots,z}$ denotes the multinomial coefficient $\frac{w!}{a!b!\cdots z!}$.

4 Let $\mathbb{F}$ be an arbitrary field. Let $n \in \mathbb{N}$ and $\lambda \vdash n$. The λ-Specht module over $\mathbb{F}$ is denoted by $\mathcal{S}^\lambda$, and the λ-Young permutation module over $\mathbb{F}$ by M^λ.

(a) Let $t \in \Delta^\lambda$. Prove that $\mathfrak{b}_t \cdot M^\lambda = \mathbb{F}e(t)$, where $\mathfrak{b}_t = \sum_{g \in C(t)} \operatorname{sgn}(g)g$.

Suppose that λ has a_j parts equal to j, for each $j \in [n]$. That is, $\lambda = (n^{a_n}, \ldots, 2^{a_2}, 1^{a_1})$.

(b) Define an equivalence relation $*$ on the set Ω^λ of λ-tabloids as follows: for $\omega, \upsilon \in \Omega^\lambda$, we say that $\omega * \upsilon$ if υ may be obtained from ω by permuting the rows of ω. For example, if $\lambda = (3, 3, 2)$ and

$$\omega = \begin{array}{l} \hline 1\,2\,3 \\ \hline 4\,5\,6 \\ \hline 7\,8 \\ \hline \end{array}\ , \quad \upsilon = \begin{array}{l} \hline 4\,5\,6 \\ \hline 1\,2\,3 \\ \hline 7\,8 \\ \hline \end{array}\ , \quad \tau = \begin{array}{l} \hline 1\,2\,4 \\ \hline 3\,5\,6 \\ \hline 7\,8 \\ \hline \end{array}\ ,$$

then $\omega * \upsilon$ but not $\omega * \tau$. Let $t, u \in \Delta^\lambda$ be any two λ-tableaux.

(i) Define $A_{tu} := \{\omega \in \Omega^\lambda \mid \langle e(t), \omega\rangle \cdot \langle e(u), \omega\rangle \neq 0\}$. Show that A_{tu} is a union of $*$-equivalence classes.

(ii) Hence, or otherwise, show that $\langle e(t), e(u)\rangle$ is a multiple of $\prod_{j=1}^{n}(a_j!)$.

From now on, suppose $\operatorname{char}(\mathbb{F}) = p > 0$.

[In parts (c) – (d) below, you may use the following facts without proof:

Fact 1 : Given any $t \in \Delta^\lambda$, there exists $u \in \Delta^\lambda$ such that $\langle e(t), e(u)\rangle = \prod_{j=1}^{n}(a_j!)^j$.

Fact 2 : Let $\lambda, \mu \vdash n$. If $t \in \Delta^\lambda$ and $v \in \Delta^\mu$ satisfy $\mathfrak{b}_t \cdot \{v\} \neq 0$, then $\lambda \trianglerighteq \mu$.]

(c) (i) Define what it means for λ to be a p-regular partition, in terms of the multiplicities $a_1, \ldots, a_n$.

(ii) Show that $\mathcal{S}^\lambda \nleqslant (\mathcal{S}^\lambda)^\perp$ if and only if λ is p-regular.

(d) Let λ, $\mu \vdash n$ and suppose λ is p-regular. Let $V \leqslant M^\mu$, and suppose that $\theta : \mathcal{S}^\lambda \to \frac{M^\mu}{V}$ is a non-zero $\mathbb{F}S_n$–homomorphism. Fix any $t \in \Delta^\lambda$.

(i) Show that there exists $u \in \Delta^\lambda$ such that $\mathfrak{b}_t \cdot e(u) = \alpha e(t)$ for some $0 \neq \alpha \in \mathbb{F}$.

(ii) Hence, or otherwise, show that $\mathfrak{b}_t \cdot \frac{M^\mu}{V} \neq 0$. Deduce that $\lambda \trianglerighteq \mu$.

(iii) Now further assume that $\lambda = \mu$. Show that $\theta(e(t))$ is a scalar multiple of $e(t) + V$. Deduce that $\dim_{\mathbb{F}} \operatorname{End}_{\mathbb{F}S_n}(\frac{\mathcal{S}^\lambda}{\mathcal{S}^\lambda \cap (\mathcal{S}^\lambda)^\perp}) = 1$.

(e) Let α and β be two p-regular partitions of n. Prove that $\frac{\mathcal{S}^\alpha}{\mathcal{S}^\alpha \cap (\mathcal{S}^\alpha)^\perp} \cong \frac{\mathcal{S}^\beta}{\mathcal{S}^\beta \cap (\mathcal{S}^\beta)^\perp}$ if and only if $\alpha = \beta$.

END OF PAPER