MAT3

MATHEMATICAL TRIPOS Part III

Friday, 2 June, 2023 9:00 am to 12:00 pm

PAPER 154

INTRODUCTION TO NON-LINEAR ANALYSIS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **BOTH** questions.
There are **TWO** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Scattering bubbles for L^2 critical defocusing (NLS)

In this question we work in $\mathbb{R}^2$ with complex valued functions. We define $\Sigma = H^1 \cap \{xu \in L^2\}$ equipped with the norm

$$\|v\|_\Sigma^2 = \int_{\mathbb{R}^2} \left(|\nabla v|^2 + |x|^2 v^2\right) dx.$$

We fix once and for all a constant $w > 2$ and define

$$J(u) = \frac{1}{2}\|v\|_\Sigma^2 - \frac{w}{2}\int_{\mathbb{R}^2} |v|^2 + \frac{1}{4}\int_{\mathbb{R}^2} |v|^4.$$

We recall that the L^2 critical defocusing (NLS) in $\mathbb{R}^2$ is

$$i\partial_t u + \Delta u - u|u|^2 = 0.$$

1) Let $\psi(x) = e^{-\frac{|x|^2}{2}}$, compute $(-\Delta + |x|^2)\psi$.

2) Show that
$$\forall \varepsilon > 0, \exists C_\varepsilon \text{ such that } \|v\|_{L^2}^2 \leqslant \varepsilon \|v\|_\Sigma^2 + C_\varepsilon \|v\|_{L^4}^2$$
and conclude that
$$J \equiv \inf_{u\in\Sigma} J(u) > -\infty.$$

3) Show that $J < 0$.

4) Show that J is attained, and that there exists a non trivial minimizer v with $v \geqslant 0$. Derive the equation satisfied by v.

5) Let $v \geqslant 0$ be the above minimizer. Let $y = \frac{x}{\lambda(t)}$. Show that
$$u(t,x) = \frac{1}{\lambda(t)}\left(v(y)e^{-i\frac{b(t)|y|^2}{4}}\right)e^{i\gamma(t)}$$
solves the L^2 critical defocusing (NLS) equation as long as the parameters satisfy the dynamical system
$$\left|\ \frac{ds}{dt} = \frac{1}{\lambda^2},\quad \frac{db}{ds} + b^2 = -4,\quad b = -\frac{1}{\lambda}\frac{d\lambda}{ds},\quad \frac{d\gamma}{ds} = -w \right. \tag{0.1}$$

6) Solve (0.1) with data $\lambda(t=0) = 1$, $b(t=0) = 0$, $\gamma(t=0) = 0$. (Hint: compute $\frac{d}{ds}\left(\frac{\sqrt{b^2+4}}{\lambda}\right)$).

7) Let $S(t)$ be the free Schrödinger semi group. Show using Strichartz that the sequence of functions
$$t \mapsto \int_{-1}^{t} S(-s)\left(u(s,y)|u(s,y)|^2\right) ds$$
has a strong L^2 limit as $t \to +\infty$ and conclude that there exists $u_\infty \in L^2$ such that
$$u - S(t)u_\infty \to 0 \text{ in } L^2 \text{ as } t \to +\infty.$$

2 Morawetz estimates for the defocusing wave equation

In this question, we work in $\mathbb{R}^3$ with real valued radially symmetric functions. We pick an odd integer $p > 1$ and let $u(t, r)$ solve the defocusing wave equation

$$\partial_{tt} u - \Delta u + u|u|^{p-1} = 0, \quad x \in \mathbb{R}^3.$$

We assume that the solution is global in time, $\mathcal{C}^\infty$ smooth in both (t, x) with the global Sobolev regularity $(u(t,\cdot), \partial_t u(t,\cdot)) \in \dot{H}^1 \times L^2$. In all questions, integration by parts in space can be performed without boundary terms, no need for any justification.

1) Show that there exists a universal constant $C > 0$ such that

$$\forall u \in H^1, \quad \int_{\mathbb{R}^3} \frac{u^2}{|x|^2} dx \leqslant C \int_{\mathbb{R}^3} |\nabla u|^2.$$

2) Show that the total energy

$$E = \int_{\mathbb{R}^3} \left[\frac{(\partial_t u)^2 + |\nabla u|^2}{2} + \frac{|u|^{p+1}}{p+1} \right] dx$$

is invariant in time.

3) Let a smooth $\psi(r)$, show the Morawetz identity

$$-\frac{d}{dt}\left\{ \int_{\mathbb{R}^3} \partial_t u \left[\frac{\Delta\psi}{2} u + \nabla\psi \cdot \nabla u \right] dx \right\} = \int_{\mathbb{R}^3} \left[\psi'' |\nabla u|^2 dx - \frac{1}{4}(\Delta^2 \psi) u^2 + \frac{p-1}{2(p+1)} (\Delta\psi) |u|^{p+1} \right] dx.$$

4) Let $\psi'(r) = r$. What is $-\Delta^2 \psi$ in $\mathcal{D}'(\mathbb{R}^3)$? Conclude that this choice of ψ implies

$$\int_0^{+\infty} \int_{\mathbb{R}^3} \frac{|u|^{p+1}}{|x|} dx dt \leqslant C E(u_0). \tag{0.1}$$

Can this equation admit finite energy stationary solutions?

5) Let $w = ru$, compute the equation satisfied by w.

6) Let a smooth $\psi(r)$ and define

$$J_\psi(t) = \int_{r>0} \psi \left[\frac{1}{2}(\partial_t w + \partial_r w)^2 + \frac{1}{p+1} \frac{|w|^{p+1}}{r^{p-1}} \right] dr$$

compute $\frac{dJ_\psi}{dt}$.

7) Conclude that the modified Morawetz holds

$$\begin{aligned} &\frac{d}{dt}\left\{ \int_{\mathbb{R}^3} \psi \left[\frac{1}{2} \left(\partial_t u + \partial_r u + \frac{u}{r} \right)^2 + \frac{1}{p+1} |u|^{p+1} \right] dx \right\} \\ = &-\frac{1}{2} \int \int_{\mathbb{R}^3} \psi' \left[\partial_t u + \partial_r u + \frac{u}{r} \right]^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \left[\psi' - (p-1)\frac{\psi}{r} \right] dx \end{aligned}$$

and give another proof of (0.1) using a suitable $\psi(r)$.

END OF PAPER