UNIVERSITY OF CAMBRIDGE

MAMA/151, NST3AS/151, MAAS/151

MAT3

MATHEMATICAL TRIPOS Part III

Monday, 12 June, 2023 9:00 am to 11:00 am

PAPER 151

GROUP COHOMOLOGY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define what is meant by an extension of a group G by a $\mathbf{Z}G$-module M. What does it mean for such an extension to be split? What does it mean for two extensions of G by M to be equivalent?

Show that any extension equivalent to a split one is also split.

Explain how to obtain an element of $H^2(G,M)$ from an extension of G by M.

Let G be a free Abelian group of rank 2. Let I be the kernel of the augmentation map $\epsilon : \mathbf{Z}G \longrightarrow \mathbf{Z}$. Let M be the $\mathbf{Z}G$-module $\mathbf{Z}G/I^2$. By using an appropriate resolution of the trivial $\mathbf{Z}G$-module $\mathbf{Z}$, calculate $H^2(G,M)$.

Why is the canonical map from $H^2(G,M)$ to $H^2(G,\mathbf{Z})$ surjective?

Deduce that the integral Heisenberg group, consisting of the strictly upper triangular 3-by-3 integral matrices, has a central non-split extension by $\mathbf{Z}^2$.

2

Let $G = F/R$ where F is a free group of rank n.

Let R_{ab} be the abelianisation of R. Describe the $\mathbf{Z}G$-module structure induced on R_{ab}.

Show that there is an exact sequence of $\mathbf{Z}G$-modules

$$0 \longrightarrow R_{ab} \longrightarrow \mathbf{Z}G^n \longrightarrow \mathbf{Z}G \longrightarrow \mathbf{Z} \longrightarrow 0$$

Let M be a $\mathbf{Z}G$-module. Prove MacLane's Theorem, that there is an exact sequence

$$H^1(F,M) \longrightarrow \mathrm{Hom}_G(R_{ab},M) \longrightarrow H^2(G,M) \longrightarrow 0.$$

Give an example where R is non-trivial and the left hand map

$$H^1(F,M) \longrightarrow \mathrm{Hom}_G(R_{ab},M)$$

is not injective.

3

Let k be a field. State the Artin-Wedderburn Theorem concerning central simple k-algebras. Show that the tensor product of two central simple k-algebras is also a central simple k-algebra. Define the Brauer group $\mathrm{Br}(k)$. [You need not check the group axioms, but you should give the identity and inverses.]

Let L be a finite Galois extension of k. Let G_L be the (finite) Galois group of the extension. Regarding the multiplicative group $L^\times$ of L as a $\mathbf{Z}G_L$-module, explain how to define the central simple k-algebra $A(L, G_L, \phi)$ arising from a normalised 2-cocycle ϕ of G_L with coefficients in $L^\times$, and show that there is a map from $H^2(G_L, L^\times)$ to $\mathrm{Br}(k)$. [You need not show that it is a group homomorphism.]

Show that for any central simple algebra A there is some tensor power of A isomorphic to $M_n(k)$ for some n. [You may assume that each element of $\mathrm{Br}(k)$ lies in the image of $H^2(G_L, L^\times)$ for some choice of extension L of k. However you should prove any results you use about $H^2(G_L, L^\times)$.]

4

Let G be a group with normal subgroup H. Let $Q = G/H$.

Give a brief sketch of the construction of the Lyndon-Hochschild-Serre spectral sequence and how to use it to calculate the cohomology $H^*(G, \mathbf{Z})$, where G acts trivially on $\mathbf{Z}$. You should calculate the cohomology $H^*(G, \mathbf{Z})$ for the dihedral group G of order 10, with normal cyclic subgroup H of order 5, to illustrate your discussion. [You may assume that in this case the induced action of the generator of Q on $H^2(H, \mathbf{Z})$ is by multiplication by -1.]

END OF PAPER