UNIVERSITY OF CAMBRIDGE

MAMA/144, NST3AS/144, MAAS/144

MAT3
MATHEMATICAL TRIPOS Part III

Tuesday, 6 June, 2023 1:30 pm to 3:30 pm

PAPER 144

MODEL THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let M be an L-structure. Let $p(\bar{x})$ be a type in $L(M)$ that is finitely satisfiable in M. Show that $p(\bar{x})$ is realised in some N such that $M \preccurlyeq N$.

(b) Let T be a complete theory with no finite models and a monster model $\mathcal{U}$. For $\mathcal{D} \subseteq \mathcal{U}$ and $f \in \mathrm{Aut}(\mathcal{U})$, define $f[\mathcal{D}] = \{f(d) : d \in \mathcal{D}\}$ and $\mathrm{O}(\mathcal{D}) = \{f[\mathcal{D}] : f \in \mathrm{Aut}(\mathcal{U})\}$.

Suppose $\mathcal{D} = \phi(\mathcal{U}, \bar{b})$ for some $\bar{b} \in \mathcal{U}^{|\bar{b}|}$. Show that if $\mathrm{O}(\mathcal{D})$ is infinite, then $|\mathrm{O}(\mathcal{D})| = |\mathcal{U}|$.

2 Let T_{rg} be the theory of the random graph in the language $L_{\mathrm{gph}} = \{R\}$, where R is a binary relation symbol. Let $\mathcal{U}$ be the monster model of T_{rg}.

(a) Show that every partial embedding $p : \mathcal{U} \to \mathcal{U}$ is elementary.

(b) Hence, or otherwise, show that T_{rg} has quantifier elimination. [You may assume that a formula $\phi(\bar{x})$ is preserved by all partial embeddings if and only if it is equivalent to a quantifier-free formula.]

(c) Let $A \subseteq \mathcal{U}$ be a small subset of $\mathcal{U}$. Define $\mathrm{acl}(A)$ and $\mathrm{dcl}(A)$.

(d) Show that, for A as in part (c), $\mathrm{acl}(A) = \mathrm{dcl}(A)$.

3 Let T be a complete theory with no finite models.

(a) Define what it means for a model N of T to be saturated, universal and homogeneous.

(b) Show that if N is universal and homogeneous then N is saturated. [You may assume any characterization of saturation, provided you state it clearly and correctly].

(c) Let N be an uncountable saturated model of T (for example a monster model), and let the formula $\phi(x, y)$ define an equivalence relation E on N. Suppose that every model $M \preccurlyeq N$ such that $|M| < |N|$ intersects every equivalence class of E. Show that E has finitely many equivalence classes.

4 Let T be a strongly minimal theory with monster model $\mathcal{U}$.

(a) Let A and B be independent subsets of $\mathcal{U}$. Show that every bijection $f : A \to B$ is an elementary map.

(b) Assume that the language of T is countable. Show that every model (that is, every $M \preceq \mathcal{U}$ such that $|M| < |\mathcal{U}|$) is homogeneous. [You may assume standard properties of algebraic closure].

END OF PAPER