MAMA/142, NST3AS/142, MAMA/142

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday, 6 June, 2023 1:30 pm to 4:30 pm

PAPER 142

CHARACTERISTIC CLASSES AND K-THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $\pi : E \to X$ be a d-dimensional real vector bundle over a compact Hausdorff space. Using the projective bundle formula, define the Stiefel–Whitney classes $w_i(E) \in H^i(X; \mathbb{Z}/2)$. If $\pi' : E' \to X$ is another real vector bundle, prove that

$$w_k(E \oplus E') = \sum_{i+j=k} w_i(E) \cdot w_j(E').$$

If $\pi : L \to X$ and $\pi' : L' \to X$ are real line bundles, prove that

$$w_1(L \otimes L') = w_1(L) + w_1(L').$$

[*You may use any description of the cohomology of* $\mathbb{RP}^n$*.*]

Let $\pi : E \to M$ be a d-dimensional real vector bundle with inner product over a smooth manifold M, $p : \mathbb{P}(E) \to M$ denote its associated projectivisation, $L_E \to \mathbb{P}(E)$ denote the tautological line bundle, and ω_E denote the orthogonal complement of $L_E \subset p^*E$. You may assume that $\mathbb{P}(E)$ is again a smooth manifold, and that its tangent bundle may be described as

$$T\mathbb{P}(E) \cong p^*TM \oplus \mathrm{Hom}(L_E, \omega_E).$$

When $M = \mathbb{RP}^n$ and E is the direct sum of k copies of the tautological line bundle $\gamma_{\mathbb{R}}^{1,n+1}$, determine the ring $H^*(\mathbb{P}(E); \mathbb{Z}/2)$ and hence describe the total Stiefel–Whitney class

$$w(T\mathbb{P}(E)) \in H^*(\mathbb{P}(E); \mathbb{Z}/2).$$

2

What is the *Bott isomorphism*? Using the Bott isomorphism, explain how to calculate $\widetilde{K}^i(S^d)$ for all $d \geqslant 0$ and all $i \in \mathbb{Z}$.

If Y is a finite CW-complex which only has even-dimensional cells, prove that $K^{-1}(Y) = 0$ and $K^0(Y)$ is a free abelian group, and hence prove that there is an isomorphism

$$K^0(Y) \otimes K^i(X) \xrightarrow{\sim} K^i(Y \times X).$$

If $f : Z \to Z$ is a continuous map, let $T_f := [0,1] \times Z/\sim$ denote its mapping torus, where $(1, z) \sim (0, f(z))$. If Z is a compact Hausdorff space such that $K^{-1}(Z) = 0$, prove that $K^0(T_f)$ is isomorphic to

$$\{x \in K^0(Z) \;:\; f^*(x) = x\}.$$

For the map $f : \mathbb{CP}^2 \times \mathbb{CP}^2 \to \mathbb{CP}^2 \times \mathbb{CP}^2$ given by $(x, y) \mapsto (y, x)$, determine $K^0(T_f)$ as an abelian group.

[*You may use any description of the K-theory of $\mathbb{CP}^2$ without proof, and any results from the course provided that they are clearly stated.*]

3

State the splitting principle in cohomology for complex vector bundles. Define the Chern character

$$ch : K^0(X) \longrightarrow H^{ev}(X; \mathbb{Q})$$

and prove carefully that it is a ring homomorphism. [*You may use any results from the theory of symmetric polynomials provided they are clearly stated, as well as the formula for the first Chern class of a tensor product of line bundles.*]

Explain why $ch : \widetilde{K}^0(S^{2n}) \to \widetilde{H}^{ev}(S^{2n}; \mathbb{Q})$ has image inside $\widetilde{H}^{ev}(S^{2n}; \mathbb{Z})$, and hence prove that for any complex vector bundle $\pi : E \to S^{2n}$, the integer $\langle c_n(E), [S^{2n}] \rangle$ is divisible by $(n-1)!$. [*Hint: You should use a relation between the power sum symmetric polynomials and the elementary symmetric polynomials.*]

4

Let $\pi : E \to X$ be a d-dimensional complex vector bundle. Denote by $\lambda_E \in \widetilde{K}^0(Th(E))$ the K-theory Thom class, and by $e^K(E) = \Lambda_{-1}(\overline{E})$ its associated Euler class. Assuming that the Thom class induces a Thom isomorphism, derive the Gysin sequence for the sphere bundle $p : \mathbb{S}(E) \to X$.

Writing $Y := \mathbb{S}(\gamma_{\mathbb{C}}^{1,n+1} \oplus \gamma_{\mathbb{C}}^{1,n+1})$, calculate $K^{-1}(Y)$ as an abelian group.

[*You may use any description of the K-theory of $\mathbb{CP}^n$.*]

Define the cannibalistic class $\rho^k(E)$ in terms of the Adams operation ψ^k, and explain why it satisfies $\rho^k(E \oplus E') = \rho^k(E) \cdot \rho^k(E')$ when $\pi' : E' \to X$ is another complex vector bundle. Derive a formula for $\rho^k(E)$ when E is a complex line bundle.

[*You may use any properties of the Thom class and the Adams operations, provided they are clearly stated.*]

For the map $p_! : K^{-1}(\mathbb{S}(E)) \to K^0(X)$ in the Gysin sequence, explain how to evaluate $p_!(\psi^k(x))$ in terms of $\psi^k(p_!(x))$.

Hence determine the action of ψ^2 on $K^{-1}(Y)$ in an appropriate basis.

END OF PAPER