UNIVERSITY OF CAMBRIDGE

MAMA/137, NST3AS/137, MAAS/137

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday, 13 June, 2023 1:30 pm to 4:30 pm

PAPER 137

MODULAR FORMS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $k \geqslant 4$ be an even integer.

(a) Show that any element $\tau \in \mathfrak{h}$ is conjugate, under the action of $\Gamma(1)$, to an element of the set

$$\mathcal{F} = \{\tau \in \mathfrak{h} \mid \mathrm{Re}(\tau) \in [-1/2, 1/2], \mathrm{Im}(\tau) \geqslant \sqrt{3}/2\}.$$

(b) Show that if $f(\tau) = \sum_{n \geqslant 1} a_n q^n \in S_k(\Gamma(1))$, then there is a constant $C > 0$ such that $|a_n| \leqslant C n^{k/2}$ for all $n \geqslant 1$.

(c) Show conversely that if $g(\tau) = \sum_{n \geqslant 0} b_n q^n \in M_k(\Gamma(1))$ and there is a constant $C > 0$ such that $|b_n| \leqslant C n^{k/2}$ for all $n \geqslant 1$, then $g \in S_k(\Gamma(1))$. [You may use the formula

$$G_k(\tau) = 2\zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

for the q-expansion of $G_k(\tau)$.]

2

Let $k \in \mathbb{Z}$, and let p be a prime number.

(a) Define what is meant by a *modular function* and a *modular form* of weight k and level $\Gamma(1)$.

(b) Let f be a modular function of weight k and level $\Gamma(1)$ which is holomorphic in $\mathfrak{h}$. Show that the function $T_p(f) : \mathfrak{h} \to \mathbb{C}$, defined by

$$T_p(f) = p^{k-1} f(p\tau) + \frac{1}{p} \sum_{b=0}^{p-1} f((\tau + b)/p),$$

is a modular function of weight k and level $\Gamma(1)$ which is holomorphic in $\mathfrak{h}$.

(c) Let f be a modular function of weight k and level $\Gamma(1)$ which is holomorphic in $\mathfrak{h}$. Suppose that the $\mathbb{C}$-vector space spanned by the functions

$$f, T_p(f), T_p^2(f), \ldots, T_p^n(f), \ldots \ (n \geqslant 1)$$

is finite-dimensional. Show that f is a modular form.

3

(a) Let p be a prime number, and define

$$\Gamma_1(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid a \equiv 1 \bmod p, c \equiv 0 \bmod p \right\}.$$

Compute the number of cusps of $\Gamma_1(p)$.

(b) Let $j(\tau) = E_4(\tau)^3/\Delta(\tau)$, $f(\tau) = j(\tau)/j(2\tau)$. Show that f is a modular function of weight 0 and level $\Gamma_1(2)$.

(c) For each cusp $\Gamma_1(2) \cdot z$ of $\Gamma_1(2)$, decide (with proof) whether $f(\tau)$ is holomorphic at $\Gamma_1(2) \cdot z$.

[You may assume any necessary properties of the modular forms E_4, Δ, providing you state them precisely.]

4

Let $k \in 2\mathbb{Z}$.

(a) Define

$$\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid c = 0 \right\},$$

and fix $s \in \mathbb{C}$ with $\mathrm{Re}(s) > (2-k)/2$. For $\tau \in \mathfrak{h}$, define

$$E_{k,s}(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} \mathrm{Im}(\gamma\tau)^s j(\gamma, \tau)^{-k}.$$

Show that $E_{k,s}(\tau)$ is absolutely and locally uniformly convergent in $\mathfrak{h}$.

(b) Let $f \in M_k(\Gamma(1))$. Show that the function $f(\tau)\overline{E_{k,s}(\tau)}\mathrm{Im}(\tau)^k$ is invariant under the (weight 0) action of $\Gamma(1)$.

END OF PAPER