UNIVERSITY OF CAMBRIDGE

MAMA/136, NST3AS/136, MAAS/136

MAT3

MATHEMATICAL TRIPOS Part III

Thursday, 1 June, 2023 9:00 am to 12:00 pm

PAPER 136

LOCAL FIELDS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let K be a non-archimedean local field and L/K a Galois extension. Define the Weil group $W(L/K)$ and describe its topology. Give an example of an L and an open subgroup $U \subset W(L/K)$ which is not open as a subgroup of $\mathrm{Gal}(L/K)$.

(b) State the main theorem of local class field theory (local Artin reciprocity), and state the Existence Theorem. Show that for L/K a finite abelian extension, we have $[N_{L/K}(\mathcal{O}_L^\times) : \mathcal{O}_K^\times] = e_{L/K}$, where $e_{L/K}$ is the ramification index of L/K.

(c) Let p be an odd prime and set $L_1 = \mathbb{Q}_p(\zeta_p)$ and $L_2 = \mathbb{Q}_p(\sqrt[p-1]{-p})$ where ζ_p is a primitive p^{th}-root of unity. Show that $L_1/\mathbb{Q}_p$ and $L_2/\mathbb{Q}_p$ are totally ramified Galois extensions and compute $N_{L_1/\mathbb{Q}_p}(L_1^\times)$ and $N_{L_2/\mathbb{Q}_p}(L_2^\times)$. Deduce that $L_1 = L_2$. [*Results about the structure of* $\mathbb{Q}_p^\times$ *may be used without proof as long as they are stated clearly.*]

2

(a) State and prove a version of Hensel's Lemma.

(b) Show that $\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2 \cong (\mathbb{Z}/2\mathbb{Z})^3$.

(c) Let p be an odd prime number. Determine the smallest integer $k \geqslant 1$ such that for $\alpha \in \mathbb{Z}_p^\times$, we have $\alpha \in (\mathbb{Z}_p^\times)^p$ if and only if $\alpha \bmod p^k \in ((\mathbb{Z}/p^k\mathbb{Z})^\times)^p$.

3 (a) Let L/K be a finite Galois extension of non-archimedean local fields. Define the higher ramification groups $G_i(L/K)$ for $i \in \mathbb{Z}_{\geqslant -1}$. Show that if L/K is a totally ramified extension and $\pi_L \in L$ is a uniformizer, then we have

$$G_i(L/K) = \{\sigma \in \mathrm{Gal}(L/K) | v_L(\sigma(\pi_L) - \pi_L) \geqslant i+1\},$$

where v_L is the normalized valuation on L. Deduce (still assuming L/K totally ramified) that there is an injective group homomorphism $G_0(L/K)/G_1(L/K) \hookrightarrow k_L^\times$, where k_L is the residue field of L.

(b) Let L be the splitting field of the polynomial $f(X) = X^3 + 3X + 3$ over $\mathbb{Q}_3$. Compute the higher ramification groups $G_i(L/\mathbb{Q}_3)$ for $i \geqslant -1$. [*The discriminant of* $X^3 + pX + q$ *is* $-4p^3 - 27q^2$.]

4 (a) State and prove Krasner's Lemma, and use it to show that every finite extension of $\mathbb{Q}_p$ arises as the completion of a number field.

(b) Let L/K be a Galois extension of number fields and $\mathfrak{p}$ a non-zero prime ideal of $\mathcal{O}_K$. Show that $\mathrm{Gal}(L/K)$ acts transitively on the set of prime ideals $\mathcal{P}$ of $\mathcal{O}_L$ with $\mathcal{P} \cap \mathcal{O}_K = \mathfrak{p}$. For $\mathcal{P} \subset \mathcal{O}_L$ such a prime ideal, define the decomposition group $G_{\mathcal{P}/\mathfrak{p}}$, and describe its relation with the Galois group $\mathrm{Gal}(L_{\mathcal{P}}/K_{\mathfrak{p}})$ of the completions.

Let $L/\mathbb{Q}$ denote the splitting field of the polynomial $X^3 - 7$. Describe the decomposition groups for all primes of $\mathcal{O}_L$ lying above $p = 7$.

5 Let $(K, |\cdot|)$ be a discretely valued field.

(a) Define the valuation ring $\mathcal{O}_K$ and show that it is an integral domain with a unique maximal ideal $\mathfrak{m}$. Show that every ideal in $\mathcal{O}_K$ is principal.

(b) Let L/K be a finite Galois extension and let $|\cdot|_L$ be an absolute value on L extending $|\cdot|$.

(i) Show that if K is complete, then $|\cdot|_L$ is the unique absolute value on L extending $|\cdot|$, and deduce that $|\sigma(x)|_L = |x|_L$, for all $x \in L$ and $\sigma \in \mathrm{Gal}(L/K)$.

(ii) Show that if K is complete, then $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in L. Give an example to show that this need not hold when K is not complete.

END OF PAPER