UNIVERSITY OF CAMBRIDGE

MAMA/133, NST3AS/133, MAAS/133

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday, 13 June, 2023 9:00 am to 11:00 am

PAPER 133

GEOMETRIC GROUP THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Prove that the subgroup

$$\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$$

of $SL_2(\mathbb{Z})$ is free of rank 2. [*Hint: Consider the sets* $A = \{(x, y) \in \mathbb{R}^2 \mid |x| > |y|\}$ *and* $B = \{(x, y) \in \mathbb{R}^2 \mid |x| < |y|\}$.]

(b) Recall that a group G is *residually finite* if, for every non-trivial element g of G, there is a homomorphism f to a finite group such that $f(g) \neq 1$. Prove that the free group of rank 2 is residually finite.

2 (a) Prove that the Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{Z} \right\}$$

can be written as a semi-direct product $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ for a certain matrix A, which you should state explicitly. Compute the abelianisation of H.

(b) Let $\Gamma_B = \mathbb{Z}^2 \rtimes_B \mathbb{Z}$, where $B \in GL_2(\mathbb{Z})$. Prove that, if 1 is not an eigenvalue of B, then the image of the $\mathbb{Z}^2$ factor in the abelianisation of Γ_B is finite. [*Hint: Consider a commutator* $tnt^{-1}n^{-1}$, *where* t *generates the* $\mathbb{Z}$ *factor and* n *is contained in the* $\mathbb{Z}^2$ *factor.*]

(c) Exhibit a matrix $C \in GL_2(\mathbb{Z})$ such that the abelianisation of any finite-index subgroup of Γ_C is virtually cyclic. Justify your answer.

3 Throughout this question, H is a subgroup of G.

(a) Consider finite generating sets S for G and T for H, and let d_S and d_T be the corresponding word metrics. Prove that

$$d_S(h_1, h_2) \leqslant C \, d_T(h_1, h_2)$$

for some constant C and all elements $h_1, h_2 \in H$.

(b) The subgroup H is said to be a *retract* if there is a homomorphism $r : G \to H$ such that $r(h) = h$ for all $h \in H$. Prove that retracts are quasi-isometrically embedded.

(c) Let

$$G = \langle a, b \mid bab^{-1} = a^2 \rangle$$

and let $H = \langle a \rangle$. Prove that H is not quasi-isometrically embedded in G.

4 (a) Let X be a δ-hyperbolic metric space and let $\gamma : \mathbb{R} \to X$ be an isometric embedding. For any $x \in X$, prove that there is $t \in \mathbb{R}$ that minimises $d(x, \gamma(t))$.

Now suppose that $t_1, t_2 \in \mathbb{R}$ both minimise $d(x, \gamma(t))$. Prove that $|t_1 - t_2| \leqslant 6\delta$.

(b) Consider a geodesic triangle in X with vertices x, y and z. Let p be the point on the side zx with

$$d(z, p) = \frac{d(x, z) + d(y, z) - d(x, y)}{2}\,.$$

Similarly, let q be the point on xy with

$$d(x, q) = \frac{d(y, x) + d(z, x) - d(y, z)}{2}\,,$$

and let r be the point on yz with

$$d(y, r) = \frac{d(z, y) + d(x, y) - d(z, x)}{2}\,.$$

Prove that p, q and r are all at distance at most 4δ from each other.

END OF PAPER