UNIVERSITY OF CAMBRIDGE

MAMA/130, NST3AS/130, MAAS/130

MAT3

MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2023 1:30pm to 3:30pm

PAPER 130

RAMSEY THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) State and prove the Hales-Jewett theorem. Deduce the extended Hales-Jewett theorem.

(ii) A subset of $\mathbb{N}$ of the form $\{a + \sum_{i \in I} x_i : I \subset [n]\}$, where a and $x_1, \ldots, x_n$ are fixed positive integers, is called a *Hilbert n-cube*. Use the extended Hales-Jewett theorem on alphabet $\{0, 1\}$ to show that, for any n, whenever $\mathbb{N}$ is finitely coloured there is a monochromatic Hilbert n-cube.

[You may not assume the Finite Sums theorem or van der Waerden's theorem.]

(iii) Give a direct proof, using induction on n (but not using any theorems from the course), that whenever $\mathbb{N}$ is finitely coloured there is a monochromatic Hilbert n-cube.

2

State and prove Rado's theorem.

[You may assume that, for any m, p, c, whenever $\mathbb{N}$ is finitely coloured there is a monochromatic (m, p, c)-set.]

What is the smallest prime p such that, in the 'last digit in base p' colouring of $\mathbb{N}$, there is no monochromatic solution to the equation $4x + 3y - 6z = 0$? Justify your answer.

Find a 10-colouring on $\mathbb{N}$ for which there is no monochromatic solution to the equation $x + 2y - 7z = 0$. By modifying your colouring, or otherwise, find a 5-colouring of $\mathbb{N}$ with this property.

3

State and prove Hindman's theorem. [You may assume that the space $\beta\mathbb{N}$ is a compact Hausdorff space, and also simple facts about ultrafilters, their quantifiers, and the operation $+$ on $\beta\mathbb{N}$.]

Let $x_1, x_2, \ldots$ be a sequence of distinct elements of the unit interval $[0, 1]$, and let $\mathcal{U}$ be a fixed ultrafilter. Explain why the sets $\{x_n : n \in A\}$, for each $A \in \mathcal{U}$, have the finite intersection property. Deduce that, for some $x \in [0, 1]$, every neighbourhood of x meets every such set. Can there be two such points x? Justify your answer.

4

Prove that every regular m-gon is Ramsey. [If you use a result about A-invariant colourings of X^n then you must prove it.]

A finite subset X of $\mathbb{R}^d$, for some d, is called *approximately Ramsey* if for every k and every $\epsilon > 0$ there exists a finite set S in $\mathbb{R}^n$, for some n, such that whenever S is k-coloured there is a monochromatic set X' that is within ϵ of being a copy of X – meaning that there is a bijection $x \mapsto x'$ from X to X' such that for all $x, y \in X$ the distance from x' to y' is within ϵ of the distance from x to y. Explain why $\{0, 1, 2\}$ is approximately Ramsey. Is every finite subset X of $\mathbb{R}^d$ (for every d) approximately Ramsey? Justify your answer. [You may assume that the product of two Ramsey sets is again Ramsey.]

END OF PAPER