UNIVERSITY OF CAMBRIDGE

MAMA/126, NST3AS/126, MAAS/126

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday, 6 June, 2023 9:00 am to 11:00 am

PAPER 126

ABELIAN VARIETIES

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $X = V/\Gamma$ be a complex torus.

(i) What is a Riemann form on X? What is a polarisation on X?

(ii) Let $V' \subset V$ be a complex subspace such that $\Gamma' = \Gamma \cap V'$ has rank equal to $2\dim_{\mathbb{C}} V'$. Show that $X' = V'/\Gamma'$ is a complex subtorus of X, and that every subtorus is of this form.

Let H be a polarisation on X. Show that it induces a polarisation on X', and that there exists a subtorus $X'' \subset X$ such that $X = X' + X''$ and $X' \cap X''$ is finite.

(iii) Let $X = \mathbb{C}^2/\Gamma$ be the complex torus for which Γ is the lattice generated by the columns of the matrix

$$\begin{pmatrix} 1 & 0 & i & \sqrt{2} \\ 0 & 1 & 0 & i\sqrt{3} \end{pmatrix}$$

Show that X contains a unique subtorus of dimension 1. Hence or otherwise show that there does not exist any polarisation on X.

2 In this question, all varieties are over a fixed algebraically closed field k.

(i) Give a definition of a group scheme over k. Show that every group scheme over k is separated. Give an example of a group scheme over a field which is not reduced.

(ii) State Mumford's Rigidity Lemma. Use it to show that if X is an abelian variety and G is any group variety, then for every morphism $f\colon X \to G$, there exists a homomorphism of group varieties $g\colon X \to G$ and $y \in G(k)$ such that $f = T_y \circ g$.

(iii) Show that (ii) can fail if X is merely required to be a group variety.

(iv) Show that the group law on an abelian variety is commutative.

(v) Let X be an abelian variety, and $X_1, X_2 \subset X$ closed subvarieties, both containing the identity $e \in X(k)$, such that the product map $p\colon X_1 \times X_2 \to X$, $p(x, y) = x + y$, is an isomorphism. Show that X_i are abelian varieties and that p is an isomorphism of group schemes.

3 In this question, all varieties are over a fixed algebraically closed field k.

(i) State the Theorem of the Cube. Deduce that if X is an abelian variety, $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, and Y is any variety, then for any triple of morphisms $f, g, h \colon Y \to X$, the invertible $\mathcal{O}_Y$-module

$$\mathcal{M}_{f,g,h} = (f+g+h)^*\mathcal{L} \otimes (f+g)^*\mathcal{L}^\vee \otimes (f+h)^*\mathcal{L}^\vee \otimes (g+h)^*\mathcal{L}^\vee \otimes f^*\mathcal{L} \otimes g^*\mathcal{L} \otimes h^*\mathcal{L}$$

is trivial. Deduce that for every x, $y \in X(k)$, $T^*_{x+y}\mathcal{L} \simeq T^*_x\mathcal{L} \otimes T^*_y\mathcal{L} \otimes \mathcal{L}^\vee$.

(ii) Define the map $\phi_\mathcal{L} \colon X(k) \to \operatorname{Pic} X$ attached to an invertible $\mathcal{O}_X$-module on the abelian variety X, and show that it is a homomorphism. Show also that if $\mathcal{M} \in \operatorname{Pic} X$ is in the image of $\phi_\mathcal{L}$ for some $\mathcal{L}$, then $\phi_\mathcal{M} = 0$.

(iii) Let $\mathcal{L}$, $\mathcal{L}'$ be invertible $\mathcal{O}_X$-modules, $x \in X(k)$. Show that

$$\phi_{\mathcal{L}\otimes\mathcal{L}'} = \phi_\mathcal{L}\phi_{\mathcal{L}'}, \quad \phi_{\mathcal{L}^\vee} = \phi_\mathcal{L}^{-1}, \quad \phi_{T_x^*\mathcal{L}} = \phi_\mathcal{L}$$

and that for every $n \in \mathbb{Z}$, $\phi_{[n]^*\mathcal{L}}(x) = \phi_\mathcal{L}(nx)^n$.

[*You may use the formula for* $[n]^*\mathcal{L}$ *without proof.*]

END OF PAPER