UNIVERSITY OF CAMBRIDGE

MAMA/123, NST3AS/123, MAAS/123

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday, 7 June, 2023 1:30 pm to 4:30 pm

PAPER 123

ALGEBRAIC NUMBER THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 This question concerns the *ideal theoretic* version of global class field theory. Let K be a number field.

1. Give the definition of a *modulus* of K.
2. State and prove the ideal theoretic version of the *decomposition law.*
3. Define the *Hilbert class field* of K and describe the Galois group of the Hilbert class field over K.
4. Let F be the Hilbert class field of K, and let $\mathfrak{p}$ be a prime ideal of K, unramified in F. Use the decomposition theorem to prove that $\mathfrak{p}$ splits completely in F if and only if $\mathfrak{p}$ is a principal ideal.
5. Let $K = \mathbb{Q}(\sqrt{-6})$. Find the Hilbert class field of K. You may use without proof that $h_K = 2$.

2 Let K be a number field.

1. Give the definition of the *Dedekind zeta function* and write down its *Euler product.*
2. State the *analytic class number formula.*
3. Give the definition of a *Dirichlet character* and its *Dirichlet L-series.*
4. Let K be a quadratic number field with discriminant d_K. Show that $\zeta_K(s)$ can be written as a product of Dirichlet L-series.
5. Next let $d_K < 0$ with $d_K \neq -3, -4$. Deduce the quadratic class number formula for the imaginary quadratic field K using the identity:

$$|L(\chi, 1)| = \frac{\pi}{|2 - \chi(2)|\sqrt{m}} \Big| \sum_{\substack{a \in (\mathbb{Z}/m\mathbb{Z})^\times \\ a < m/2}} \chi(a) \Big| \quad \text{if } \chi(-1) = -1.$$

   where χ is a Dirichlet character mod m.
6. Compute the class number of $\mathbb{Q}(\sqrt{-11})$.

3 Let K be a number field. Let L/K be a finite Galois extension, and write $\mathcal{S}_{L/K}$ for the prime ideals of K that split completely in L.

1. Give the definition of *Dirichlet density*.
2. State the *Chebotarev density theorem*.
3. If L/K is abelian, and if $\mathfrak{m}$ is a modulus divisible by the conductor of L/K, prove that the Artin map is surjective.
4. Compute the Dirichlet density of $\mathcal{S}_{L/K}$.
5. Compute the density of principal prime ideals of K.

4 Let K be a number field and let L/K be a finite extension.

1. (a) Give the definition of an *adele* of K and show the set of adeles forms the *ring* of adeles $\mathbb{A}_K$ (with appropriate operations).
   (b) Give the definition of the *group of ideles* $\mathbb{I}_K$.
   (c) Let $\varphi : \mathbb{A}_K^\times \to \mathbb{A}_K \times \mathbb{A}_K$ be the embedding defined by $\alpha \mapsto (\alpha, \alpha^{-1})$. Show the restricted product topology on $\mathbb{I}_K$ equals the subspace topology on $\varphi(\mathbb{A}_K^\times) \subset \mathbb{A}_K \times \mathbb{A}_K$.
2. (a) Show that we have an injective homomorphism $K^\times \hookrightarrow \mathbb{I}_K$, and show that $K^\times$ is a *discrete* subgroup of $\mathbb{I}_K$.
   (b) Show that we have an injective homomorphism $i_{L/K} : \mathbb{I}_K \hookrightarrow \mathbb{I}_L$.
   (c) Show that the norm $N_{L/K}$ of a principal idele is principal.
   (d) Prove that the norm group $N_{L/K}C_L$ of a Galois extension L/K is the norm group of an abelian extension of K.

END OF PAPER