UNIVERSITY OF CAMBRIDGE

MAMA/122, NST3AS/122, MAAS/122

MAT3
MATHEMATICAL TRIPOS Part III

Thursday, 1 June, 2023 9:00 am to 11:00 am

PAPER 122

EXTREMAL AND PROBABILISTIC COMBINATORICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State Harris's inequality, or the FKG inequality, making sure to define the relevant notions. State and prove Janson's inequalities.

(b) Let $G \sim G(n,p)$. Show that

$$\mathbb{P}(G \not\supset K_3) = \begin{cases} e^{-\Theta(p^3n^3)} & \text{if } 0 < p \leqslant n^{-1/2}; \\ e^{-\Theta(pn^2)} & \text{if } n^{-1/2} \leqslant p < 1/2. \end{cases}$$

(c) Show there exists a $C > 0$ for which the following holds. If $p = Cn^{-3/4}$ and $G \sim G(n,p)$, then G contains at least $n/10$ vertex disjoint copies of C_4, with high probability.

2

(a) For $s, k \geqslant 1$ and a graph G, define what it means for $R \subseteq V(G)$ to be (s,k)-*rich*.

Now let G be a graph with $|G| = n$ and $e(G) = m$. Let $r, s, k, t \geqslant 1$ be such that

$$\frac{(2m)^t}{n^{2t-1}} - \binom{n}{s}\left(\frac{k}{n}\right)^t \geqslant r.$$

Show that there exists a (s,k)-rich set $R \subseteq V(G)$, with $|R| \geqslant r$.

(b) Let H be a bipartite graph with bipartition $A \cup B$ where every vertex $x \in B$ has $\deg(x) \leqslant s$. Let G be a graph for which there exists exists a $(s, |H|)$-rich set $R \subseteq V(G)$ with $|R| \geqslant |A|$. Show that $G \supseteq H$.

(c) Let H be a graph. The Ramsey number $r(H)$ is the smallest n for which every 2-colouring of the edges of K_n contains a monochromatic copy of H. In other words, $r(H)$ is the minimum n so that every partition $E(K_n) = G_1 \cup G_2$, has the property that either $G_1 \supset H$ or $G_2 \supset H$.

We define Q_d to be the *hypercube graph* in dimension d. Here $V(Q_d) = \{0,1\}^d$ and two vertices $x, y \in \{0,1\}^d$ are adjacent when x, y differ in exactly one coordinate. Show that

$$r(Q_d) \leqslant 2^{4d}.$$

3

(a) State Markov's inequality and state Chebychev's inequality.

Now show that if $p \gg n^{-1}$ and $G \sim G(n,p)$ then

$$\lim_{n\to\infty} \mathbb{P}(G \text{ contains at least 100 triangles}) = 1.$$

[You may assume Markov and Chebychev without proof]

(b) Let $P = P_n$ be a monotone graph property. Define what it means for a function $p^*(n)$ to be a *threshold function* for P.

(c) Let $P = P_n$ be a monotone graph property and let $p(n) \in (0,1)$ be a sequence for which $p(n) = o(1)$ and

$$\mathbb{P}_{p(n)}(G \text{ satisfies P}) = 1/10$$

for all n, where $G \sim G(n,p)$. Show that $p(n)$ is a threshold function for P.

(d) For this part you may assume, without proof, the following theorem:

If $p \geqslant 10(\log n)/n$ then $G \sim G(n,p)$ contains a Hamiltonian cycle with high probability.

Show that there exists a $C > 0$ so that the following holds. If $p \geqslant C(\log n)/n$ then G contains a cycle of length ℓ for each $3 \leqslant \ell \leqslant n$, with high probability.

4

(a) State the regularity lemma. Make sure to state the definition of a ε-uniform pair carefully.

(b) State and prove the triangle embedding lemma.

(c) Let $p \in (0,1)$ and $\varepsilon > 0$ be fixed and let $G \sim G(n,p)$. Show that, with high probability,

$$(p-\varepsilon)|A||B| \leqslant e(A,B) \leqslant (p+\varepsilon)|A||B|$$

for all disjoint $A, B \subseteq V(G)$ with $|A|, |B| \geqslant n/\log n$.

[You may use Chernoff's inequality, without proof, so long as you state it. You may also use Markov's inequality without proof]

(d) Let $p \in (0,1)$ be fixed and let $G \sim G(n,p)$. Show that, for all $\varepsilon > 0$, the largest triangle-free subgraph of G has at most $(1+\varepsilon)pn^2/4$ edges, with high probability.

[You may use Turán's theorem without proof]

END OF PAPER