UNIVERSITY OF CAMBRIDGE

MAMA/119, NST3AS/119, MAAS/119

MAT3

MATHEMATICAL TRIPOS Part III

Monday, 12 June, 2023 1:30 pm to 4:30 pm

PAPER 119

CATEGORY THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **SIX** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain carefully what is meant by a *representation* of a functor $F : \mathcal{C} \to \mathbf{Set}$, where $\mathcal{C}$ is locally small. Prove that representations are unique up to isomorphism if they exist.

Define the 'arrow category' $(B \downarrow F)$ where $F : \mathcal{C} \to \mathcal{D}$ is a functor and B is an object of $\mathcal{D}$, and state a criterion in terms of these categories for the existence of a left adjoint to F.

If $\mathcal{C}$ is locally small, show that $F : \mathcal{C} \to \mathbf{Set}$ is representable if and only if $(1 \downarrow F)$ has an initial object, where 1 denotes a singleton set. Hence show that if $\mathcal{C}$ is also cocomplete, then $F : \mathcal{C} \to \mathbf{Set}$ is representable if and only if it has a left adjoint.

Give an example of a complete locally small category $\mathcal{C}$ and a representable functor $\mathcal{C} \to \mathbf{Set}$ which does not have a left adjoint. [*Hint*: a representable functor preserves all limits which exist in its domain.]

2 Define a *balanced* category. If $F : \mathcal{C} \to \mathcal{D}$ is a faithful functor and $\mathcal{C}$ is balanced, prove that F reflects isomorphisms.

Let $(F : \mathcal{C} \to \mathcal{D} \dashv G : \mathcal{D} \to \mathcal{C})$ be an adjunction with unit η and counit ϵ. Show that F is faithful if and only if η is a pointwise monomorphism.

Now suppose that $\mathcal{C}$ is balanced, and that every morphism of $\mathcal{D}$ factors as a strong epimorphism followed by a monomorphism. Show that η and ϵ are both pointwise monic if and only if F is full and faithful and its image is closed under strong quotients in $\mathcal{D}$ (that is, if $FA \twoheadrightarrow B$ is a strong epimorphism, then $B \cong FA'$ for some A').

By considering a suitable non-balanced category, give an example of an adjunction whose unit and counit are both pointwise monic but whose left adjoint is not full.

3 Define the terms *diagram*, *cone* over a diagram and *limit* of a diagram. Show that if a category has small products and equalizers then it has all small limits.

A functor $F : I \to J$ between small categories is called *initial* if, for every object j of J, the category $(F \downarrow j)$ is (nonempty and) connected. If F is initial, show that for any diagram $D : J \to \mathcal{C}$ the functor which sends $(A, (\gamma_j \mid j \in \text{ob } J))$ to $(A, (\gamma_{Fi} \mid i \in \text{ob } I))$ is an isomorphism from the category of cones over D to that of cones over DF. Deduce that if $\mathcal{C}$ has limits of shape I then it also has limits of shape J, and the diagram

$$\begin{array}{ccc} [J, \mathcal{C}] & \xrightarrow{\quad F^* \quad} & [I, \mathcal{C}] \\ & {\scriptstyle \lim_J} \searrow \quad \swarrow {\scriptstyle \lim_I} & \\ & \mathcal{C} & \end{array}$$

commutes up to isomorphism. Conversely, if this diagram commutes for $\mathcal{C} = \mathbf{Set}^{\text{op}}$, show that F is initial.

4 Let $F\colon \mathbf{Set} \to \mathbf{Set}$ be a functor preserving all finite limits and colimits. Show that there is a unique natural transformation $\alpha\colon 1_{\mathbf{Set}} \to F$, and that it is (pointwise) monic. Show also that F preserves countable coproducts. [*Hint*: consider pullback diagrams of the form

$$\begin{array}{ccc} A_n & \longrightarrow & \sum_{n\in\mathbb{N}} A_n \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\;n\;} & \mathbb{N} \end{array} \qquad ;$$

you may assume that $\mathbb{N}$ is characterized up to isomorphism in **Set** by the existence of diagrams

$$1 \xrightarrow{\;0\;} \mathbb{N} \xleftarrow{\;s\;} \mathbb{N} \qquad \text{and} \qquad \mathbb{N} \underset{1}{\overset{s}{\rightrightarrows}} \mathbb{N} \longrightarrow 1$$

which are respectively a coproduct and a coequalizer.]

Now suppose A is any set for which α_A is not bijective. Show that there is a countably complete non-principal ultrafilter on A (that is, a family U of subsets of A satisfying (i) $A' \in U, A' \subseteq A'' \Rightarrow A'' \in U$, (ii) for every A', just one of A' and $A \setminus A'$ is in U, (iii) U is closed under countable intersections, and (iv) no finite sets are in U). [*Hint*: given any $x \in FA$ not in the image of α_A, consider those A' for which x is in the image of $FA' \to FA$.]

Conversely, if A is any set supporting such an ultrafilter U, show that the functor $F : \mathbf{Set} \to \mathbf{Set}$ defined by $FB = B^A/\sim_U$ (where the equivalence relation $\sim_U$ identifies f and $g\colon A \rightrightarrows B$ if and only if $\{a \in A \mid f(a) = g(a)\}$ is in U) preserves finite limits and countable coproducts, and is not isomorphic to the identity functor.

5 Explain what is meant by an *exponentiable* object in a category with finite products, and show that the class of exponentiable objects is closed under finite products.

Let **Met** denote the category of metric spaces and nonexpansive maps (that is, functions $f : X \to Y$ satisfying $d(f(x), f(y)) \leqslant d(x, y)$ for all $x, y \in X$), and write $\mathbf{Met}_b$ for the full subcategory of bounded spaces (those in which the metric takes values in some finite interval $[0, R] \subseteq \mathbb{R}$). Show that **Met** and $\mathbf{Met}_b$ have finite products. [*Hint*: consider the smallest metric on $X \times Y$ that makes the projections to X and Y nonexpansive.]

If X and Y are bounded metric spaces, we define a 'distance function' $\overline{d}$ on the set $[X, Y]$ of nonexpansive maps $X \to Y$ by

$$\overline{d}(f, g) = \sup \{d(f(x), g(y)) \mid x, y \in X, d(x, y) < d(f(x), g(y))\} .$$

Show that if $\overline{d}$ is a metric, then it makes $[X, -]$ into a functor right adjoint to $(-) \times X : \mathbf{Met}_b \to \mathbf{Met}_b$.

A metric space X is called *interpolating* if, whenever we have $x, y \in X$ with $d(x, y) = r + s$, we can find $z \in X$ with $d(x, z) = r$ and $d(z, y) = s$. Show that if X is interpolating then the function $\overline{d}$ defined above satisfies the triangle inequality, and deduce that interpolating spaces are exponentiable in $\mathbf{Met}_b$.

6 Define the notion of a *local operator* j in a topos $\mathcal{E}$, and explain what is meant by the terms *j-dense monomorphism* and *j-sheaf*. Also define the closed-subobject classifier Ω_j, and prove that it is a j-sheaf.

Show that the following conditions on a local operator j are equivalent:

(i) The reflector $L : \mathcal{E} \to \mathbf{sh}_j(\mathcal{E})$ preserves the subobject classifier.

(ii) The canonical epimorphism $\Omega \twoheadrightarrow \Omega_j$ is *j-codense*, i.e. the reflector $L : \mathcal{E} \to \mathbf{sh}_j(\mathcal{E})$ maps it to an isomorphism.

(iii) The canonical monomorphism $\Omega_j \rightarrowtail \Omega$ is j-dense.

(iv) Every monomorphism $A' \rightarrowtail A$ in $\mathcal{E}$ can be factored (not necessarily uniquely) as $A' \rightarrowtail A'' \rightarrowtail A$, where $A' \rightarrowtail A''$ is j-closed and $A'' \rightarrowtail A$ is j-dense.

END OF PAPER