UNIVERSITY OF CAMBRIDGE

MAMA/118, NST3AS/118, MAAS/118

MAT3

MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2023 1:30 pm to 4:30 pm

PAPER 118

COMPLEX MANIFOLDS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define an *almost complex structure* J arising from the atlas of complex local coordinates on a complex manifold X. Show that J is well-defined independent of the choice of local coordinates.

Define the *differential forms of type* (p, q) on X and the operators ∂ and $\bar{\partial}$ for complex differential forms. Show directly from the definitions that $\bar{\partial}\,\bar{\alpha} = \overline{\partial\alpha}$ for all complex differential forms α.

Recall that $d^c = i(\bar{\partial} - \partial)$. Show that $d^c = J^{-1}dJ$.

What is a *holomorphic vector field* on a complex manifold? Let φ_0 be a projection sending each vector $z = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}$ with $z_0 \neq 0$ to the intersection of the complex line $z_0{:}z_1{:}\ldots{:}z_n$ with $U_0 = \{z \in \mathbb{C}^{n+1} \,|\, z_0 = 1\}$. For $a \in \mathbb{C}^{n+1}$ with $a_0 \neq 0$, show that

$$(d\varphi_0)_a\Big(\frac{\partial}{\partial z_j}\Big) = \begin{cases} \dfrac{1}{a_0}\dfrac{\partial}{\partial z_j} & \text{if } j \neq 0 \\ \dfrac{-1}{a_0^2}\displaystyle\sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k} & \text{if } j = 0. \end{cases}$$

Show that if $\xi(z)$ is a linear homogeneous function, then the image under $d\phi_0$ of a vector field $\xi(z)\partial/\partial z_j$ on $\mathbb{C}^{n+1}$ extends to a well-defined holomorphic vector field on $\mathbb{C}P^n$. By considering an appropriate vector field on $\mathbb{C}^{n+1}$, or otherwise, show that $\mathbb{C}P^n$ admits a holomorphic vector field vanishing only at finitely many points.

2 Define the terms *irreducible hypersurface* and *local defining function* of a hypersurface, explaining why a local defining function exists at each point. What is a *divisor* on a complex manifold? Explain what is meant by the holomorphic line bundle $[D]$ associated to a divisor D. You should state clearly the auxiliary properties of local rings of holomorphic functions that you require.

Show that $K_{\mathbb{C}P^n} \cong [-(n+1)H]$, where $H \subset \mathbb{C}P^n$ is a hyperplane.

Let V be a connected complex submanifold of $\mathbb{C}P^n$ given by the vanishing of a homogeneous polynomial p on $\mathbb{C}^{n+1}$ with $\deg p = k > 0$. Suppose that $(dp)_z \neq 0$ whenever $p(z) = 0$ and $z \neq 0$. Determine the canonical bundle K_V in terms of an appropriate divisor on V.

Let S be a compact connected Riemann surface with $P, Q \in S$ two distinct points and $[P]$, $[Q]$ the respective holomorphic line bundles over S. Let s_P and s_Q be holomorphic sections of $[P]$ and $[Q]$, respectively, such that the divisors of these sections are $(s_P) = P$ and $(s_Q) = Q$. Show that if the holomorphic line bundles $[P]$ and $[Q]$ are isomorphic then $x \in S \to s_P(x) : s_Q(x) \in \mathbb{C}P^1$ is a well-defined biholomorphic map.

[*The adjunction formula can be assumed if accurately stated.*]

3 Let X be a Hermitian manifold. What is a *real* $(1,1)$*-form* on X? What is a *positive real* $(1,1)$*-form* on X?

Let $L \to X$ be a holomorphic line bundle with Hermitian inner product on the fibres. Define the terms *holomorphic local trivialization* of L, *unitary connection* on L, *the Chern connection* on L. If A is the Chern connection on L, show that $iF(A)$ is a real $(1,1)$-form, where $F(A)$ denotes the curvature form of A.

Let $\widehat{L}$ be another holomorphic line bundle over X with $\widehat{A}$ a connection on $\widehat{L}$. Explain carefully what is meant by the induced connection $A \otimes \widehat{A}$ on $L \otimes \widehat{L}$. Show that if $iF(A)$ and $iF(\widehat{A})$ are positive real $(1,1)$-forms, then the form $iF(A \otimes \widehat{A})$ is a positive real $(1,1)$-form too.

Show that if φ and ψ are positive real $(1,1)$-forms on X and $\dim_{\mathbb{C}} X \geqslant 2$, then $(\varphi \wedge \psi)(\xi, \eta, \bar{\xi}, \bar{\eta})$ is positive at all points $x \in X$ and all linearly independent pairs of complex tangent vectors ξ, η of type $(1,0)$ at x.

[*Standard properties of connections on complex vector bundles over smooth manifolds may be assumed if accurately stated.*
If needed, you may assume that if A and B are Hermitian matrices and some real linear combination of A and B is positive definite, then there is a non-singular matrix C such that $\bar{C}^t AC$ and $\bar{C}^t BC$ are diagonal.]

4 Let X be a compact Kähler manifold. Define the *Hodge* $*$*-operator* for complex differential forms on X, explaining briefly the auxiliary concepts you require. Define the operators d^*, $\bar{\partial}^*$, the *Laplacian* $\Delta = \Delta_d$ and the *complex Laplacian* $\Delta_{\bar{\partial}}$. Show that $\Delta = 2\Delta_{\bar{\partial}}$.

Show that if α is a $\Delta_{\bar{\partial}}$-harmonic differential form on X, then the form $\alpha \wedge \omega^k$ is again $\Delta_{\bar{\partial}}$-harmonic for all $k = 1, 2, \ldots$, where ω is the Kähler form on X.

State the Hodge decomposition theorem for (p,q)-forms on a Hermitian manifold.

Let η be a $\bar{\partial}$-exact (p,q)-form on X. Show that $\eta = \bar{\partial}\bar{\partial}^*\beta$ for some (p,q)-form β. If η is also ∂-closed, prove that $\partial\bar{\partial}^*\beta$ is harmonic and that $\bar{\partial}^*\beta$ is ∂-closed. Finally, show that such an η can be expressed as $\eta = \bar{\partial}\partial\phi$, for some $\phi \in \Omega^{p-1,q-1}(X)$.

[*You can assume that $\bar{\partial}^*$ is the formal L^2-adjoint of $\bar{\partial}$ on a Hermitian manifold. You can also assume the identity $[\Lambda, \bar{\partial}] = -i\partial^*$ on Kähler manifolds if you define what Λ is.*]

END OF PAPER