UNIVERSITY OF CAMBRIDGE

MAMA/116, NST3AS/116, MAAS/116

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday, 6 June, 2023 9:00 am to 11:00 am

PAPER 116

LARGE CARDINALS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
Question 1 carries 40 marks; questions 2 and 3 carry 30 marks each.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\kappa < \lambda$ where κ is measurable and λ is inaccessible. Let U be a κ-complete nonprincipal ultrafilter on κ. Let $N := (\mathbf{V}_\lambda)^\kappa/U$ be the ultrapower of $\mathbf{V}_\lambda$ modulo U with its elementary embedding $x \mapsto [\mathrm{const}_x]$.

[*Throughout this question, you may assume without proof that this map is an elementary embedding and that* $\mathbf{V}_{\kappa+1} \subseteq M$ *for the set* M *constructed in (a). Furthermore, you may assume basic absoluteness claims without proof, provided that you state them precisely and correctly.*]

(a) Explain how to construct a transitive set $M \subseteq \mathbf{V}_\lambda$ and an elementary embedding $j \colon \mathbf{V}_\lambda \to M$. For $f \in (\mathbf{V}_\lambda)^\kappa$, give a definition of (f); define the map j and explain why it is elementary.

[*You may use without proof all claims relevant for the construction that were proved in the lectures, provided that you state them precisely and correctly.*]

(b) Define what the *critical point of an elementary embedding between models of set theory* is and prove that κ is the critical point of the embedding j from (a).

(c) Explain in the context of the elementary embedding j from (a) what a *reflection argument* is and what it means that a property of κ *reflects below* κ. Give an example of a property of κ that reflects below κ in this context and prove your claim.

(d) State the *Keisler Extension Property* and prove that an inaccessible cardinal with the Keisler Extension Property cannot be the least inaccessible cardinal.

2 Let T and S be first-order theories extending ZFC.

(a) Define $T \leqslant_{\mathrm{Cons}} S$ and $T <_{\mathrm{Cons}} S$ in the base theory ZFC.

(b) Define a sequence of theories $\{T_i\,;\, i \in \mathbb{N}\}$ and a theory T_∞ such that $T_0 = \mathsf{ZFC} + \mathsf{IC}$ and (assuming the consistency of all involved theories)

$$T_0 <_{\mathrm{Cons}} T_1 <_{\mathrm{Cons}} T_2 <_{\mathrm{Cons}} \cdots <_{\mathrm{Cons}} T_\infty.$$

Justify your claim.

[*You may use theorems proved in the lectures, provided that you state them precisely and correctly.*]

3 Let $\kappa < \lambda$ where κ is measurable and λ is inaccessible.

(a) Construct a transitive set M with $\kappa \in M$ and $|M| = \kappa$ such that

$$(M, \in) \models \mathsf{ZFC} + \text{“}\kappa \text{ is measurable”}.$$

(b) Is the property “α is a cardinal” absolute between M and $\mathbf{V}_\lambda$?

(c) Is it possible to find an M as in (a) such that the property “α is inaccessible” is absolute between M and $\mathbf{V}_\lambda$?

Justify your claims for all parts of the question.

[*You may use the Tarski-Vaught test for elementarity without proof. Also, you may use basic absoluteness statements without proof, provided you state them precisely and correctly.*]

END OF PAPER