UNIVERSITY OF CAMBRIDGE

MAMA/115, NST3AS/115, MAAS/115

MAT3

MATHEMATICAL TRIPOS Part III

Monday, 5 June, 2023 9:00 am to 12:00 pm

PAPER 115

DIFFERENTIAL GEOMETRY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let X and Y be smooth manifolds of dimensions n and m respectively, and let $F : X \to Y$ be a smooth map. Given a vector field w on Y, a *lift* of w to X is a vector field v on X such that for all $p \in X$ we have $D_pF(\mathsf{v}(p)) = \mathsf{w}(F(p))$.

(a) Define what it means for F to be a *submersion*. State and prove a result describing the local form of submersions in coordinates, and deduce that if F is a submersion then for all $q \in Y$ the subset $F^{-1}(q) \subset X$ is a submanifold.

(b) Show that if F is a submersion then for every vector field w on Y there exists a lift v of w to X, which may be chosen to satisfy $\mathsf{v}(p) = 0$ whenever $\mathsf{w}(F(p)) = 0$.

(c) For each point $q_1 \in Y$, show that there exists a neighbourhood U of q_1 with the following property: for all points $q_2 \in U$ there exists a compactly supported vector field w on Y whose flow Ψ^t satisfies $\Psi^1(q_1) = q_2$.

Now assume that F is a submersion and is proper, meaning that preimages of compact sets are compact. Define an equivalence relation $\sim$ on Y by setting $q_1 \sim q_2$ if and only if the submanifolds $F^{-1}(q_1)$ and $F^{-1}(q_2)$ are diffeomorphic.

(d) Show that the equivalence classes of $\sim$ are open, and conclude that if Y is connected then for all $q_1, q_2 \in Y$ the submanifolds $F^{-1}(q_1)$ and $F^{-1}(q_2)$ are diffeomorphic. [You may assume without proof that compactly supported vector fields are complete.]

(e) Show that this conclusion may fail if we drop the condition that F is proper. Show that it may also fail if instead we weaken the condition that F is a submersion to 'for all $q \in Y$ the subset $F^{-1}(q) \subset X$ is a submanifold'.

2 Let X be a smooth manifold.

(a) Write down an expression for the exterior derivative $\mathrm{d} : \Omega^*(X) \to \Omega^{*+1}(X)$ in local coordinates and show that it squares to zero and satisfies the graded Leibniz rule. Show that d commutes with pullback on 0-forms, and hence on r-forms for any r.

(b) Define the de Rham cohomology $H^*_{\mathrm{dR}}(X)$. Show that if $F : X \to Y$ is a smooth map then pullback F^* induces a well-defined map $H^*_{\mathrm{dR}}(Y) \to H^*_{\mathrm{dR}}(X)$. Show further that if $F_0, F_1 : X \to Y$ are smoothly homotopic then the induced maps on H^*_{dR} agree. Deduce that if F is a homotopy equivalence then the induced map on H^*_{dR} is an isomorphism. [You may assume Cartan's magic formula without proof.]

Now consider complex projective space $\mathbb{CP}^n$, and for each $i \in \{0, \dots, n\}$ let

$$U_i = \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n : z_i \neq 0\}$$

be the standard open set. Let $U = U_0$ and $V = U_1 \cup \cdots \cup U_n$.

(c) By considering the restriction of forms to U and V, prove by induction on n that $H^i_{\mathrm{dR}}(\mathbb{CP}^n) = 0$ for all odd i. [You may assume that $H^i_{\mathrm{dR}}(S^{2n-1}) = 0$ for $i \neq 0, 2n-1$. You may *not* use any results from outside the course.]

3 Let $\pi : E \to B$ be a vector bundle of rank k.

(a) In terms of local connection 1-forms, define what is meant by a *connection* $\mathcal{A}$ on E, its associated *exterior covariant derivative operator* $\mathrm{d}^{\mathcal{A}}$, and its *curvature* F. Show that $(\mathrm{d}^{\mathcal{A}})^2\sigma = F \wedge \sigma$ for any E-valued form σ.

Now suppose that E is equipped with a connection $\mathcal{A}$. Let $\mathcal{A}^\vee$ and $\mathrm{End}(\mathcal{A})$ be the induced connections on $E^\vee$ and $\mathrm{End}(E)$ respectively.

(b) Write down expressions for the exterior covariant derivative operators $\mathrm{d}^{\mathcal{A}^\vee}$ and $\mathrm{d}^{\mathrm{End}(\mathcal{A})}$ in trivialisations. State and prove the Bianchi identity for F. State and prove the appropriate Leibniz rule relating $\mathrm{d}^{\mathcal{A}}$ and $\mathrm{d}^{\mathrm{End}(\mathcal{A})}$ acting on sections of E and $\mathrm{End}(E)$.

(c) Briefly define the *parallel transport* map $\mathcal{P}_\gamma^{\mathcal{A}} : E_{\gamma(0)} \to E_{\gamma(1)}$ associated to a path $\gamma : [0, 1] \to B$. By considering the reversed path show that this map is an isomorphism. Find and prove an expression for $\mathcal{P}_\gamma^{\mathrm{End}(\mathcal{A})} : \mathrm{End}(E_{\gamma(0)}) \to \mathrm{End}(E_{\gamma(1)})$ in terms of $\mathcal{P}_\gamma^{\mathcal{A}}$.

(d) Suppose that B is path-connected and that $\mathrm{End}(E)$ admits a global section μ which is horizontal with respect to $\mathrm{End}(\mathcal{A})$. Assuming that $\mu(b) \in \mathrm{End}(E_b)$ is an isomorphism for some $b \in B$, show that it is an isomorphism for all b.

4 Let (X, g) be a Riemannian manifold and let ∇ be an arbitrary connection on TX. As usual, we write the components of the local connection 1-forms in coordinate trivialisations as ${\Gamma^i}_{jk}$.

(a) Define the *solder form* θ and *torsion* T of ∇. Hence express, with proof, what it means for ∇ to be torsion-free in terms of the ${\Gamma^i}_{jk}$.

(b) Define what it means for ∇ to be *orthogonal*. By considering $\nabla(g(\mathsf{u}, \mathsf{v}))$ for arbitrary vector fields u and v, and applying a suitable Leibniz rule, show that ∇ is orthogonal if and only if

$$\Gamma_{jki} + \Gamma_{kji} = \frac{\partial g_{jk}}{\partial x^i}.$$

Now assume that (X, g) is compact and oriented.

(c) Define the inner product $\langle \cdot, \cdot \rangle_X$ and *codifferential* δ on $\Omega^*(X)$ in terms of the Hodge star operator $\star$ and its inverse. Show that δ is adjoint to d.

(d) Define what it means for a form α to be *harmonic* and show that this holds if and only if α is closed and coclosed. State without proof the relationship between $H^p_{\mathrm{dR}}(X)$ and the space $\mathcal{H}^p(X)$ of harmonic p-forms on (X, g).

Finally, take X to be the n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$, with local coordinates $x^1, \ldots, x^n$ induced from standard Euclidean coordinates on $\mathbb{R}^n$. Equip X with the metric $\sum_i (\mathrm{d}x^i)^2$ and orientation $\partial_{x^1} \wedge \cdots \wedge \partial_{x^n}$.

(e) Write down the action of $\star$ on $\mathrm{d}x^i$. Show that if a 1-form $\alpha = \alpha_i \, \mathrm{d}x^i$ is harmonic then each α_i is harmonic. Hence show that $H^1_{\mathrm{dR}}(T^n) \cong \mathbb{R}^n$.

END OF PAPER