UNIVERSITY OF CAMBRIDGE

MAMA/114, NST3AS/114, MAAS/114

MAT3

MATHEMATICAL TRIPOS Part III

Friday, 2 June, 2023 1:30 pm to 4:30 pm

PAPER 114

ALGEBRAIC TOPOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Exhibit each of the following spaces as a finite cell complex. [You do not need to give a detailed proof.] Using the cellular chain complex or otherwise, compute their homology with coefficients in $\mathbb{Z}$ and $\mathbb{Z}/2$.

1. Real projective space $\mathbb{RP}^n$.
2. The space $T^2/\sim$, where $\sim$ is the smallest equivalence relation containing $(z_1, z_2) \sim (-z_1, \overline{z_2})$ for all $(z_1, z_2) \in S^1 \times S^1$. [Here $\overline{z}$ denotes the complex conjugate of z.]
3. The space $T^3/\sim$, where $\sim$ is the smallest equivalence relation containing $(z_1, z_2, z_3) \sim (-z_1, \overline{z}_2, \overline{z}_3)$ for all $(z_1, z_2, z_3) \in S^1 \times S^1 \times S^1$.

2 Suppose $A \subset U \subset X$. If the inclusion map $i : A \to U$ is a homotopy equivalence, prove $H_*(X, A) \cong H_*(X, U)$.

State the *excision property* for singular homology. Define what is meant by a *good pair* and state the *collapsing a pair theorem.* Taking the excision property as given, prove the collapsing a pair theorem.

Assume $n \geqslant 2$. If $f : (D^n, \partial D^n) \to (D^n, \partial D^n)$, define the *degree* of f. Show that $\deg f = \deg f|_{\partial D^n}$. If $g : (D^n \times I, \partial(D^n \times I)) \to (D^n \times I, \partial(D^n \times I))$ is given by $g(x, t) = (f(x), t)$, show that $\deg g = \deg f$. [If you use any results about the homology or cohomology of products, you must prove them.]

3 Let R be a commutative ring. If $\alpha \in C^k(X; R)$ and $\beta \in C^l(X; R)$, define their cup product $\alpha \cup \beta \in C^{k+l}(X; R)$. If $\alpha \in C^k(X, A; R)$, show that $\alpha \cup \beta \in C^{k+l}(X, A; R)$.

If $a \in H^*(X, A; R)$ and $b \in H^*(Y; R)$, define their exterior product $a \times b$. State conditions under which the map $\Phi : H^*(X, A; R) \otimes H^*(Y; R) \to H^*(X \times Y, A \times Y; R)$ given by $\Phi(a \otimes b) = a \times b$ is an isomorphism. By considering $\mathbb{RP}^2 \times \mathbb{RP}^2$ or otherwise, show that Φ is not always an isomorphism.

If $\Delta = \{(x, x) \,|\, x \in T^2\} \subset T^2 \times T^2 = T^4$, compute the cohomology ring $H^*(T^4 \setminus \Delta)$.

4 Define what it means for a vector bundle to be *R-oriented*, where R is a commutative ring. If E is an R-oriented vector bundle, define its *Euler class.* State the Thom isomorphism theorem and derive the Gysin sequence from it. Explain [with proof] how the Gysin sequence and Euler class are related.

Suppose $M^m \subset S^n$ is a smooth m-dimensional submanifold of S^n, where $n > 2m+1$. Let V be a tubular neighborhood of M. Express $H^*(\partial V; \mathbb{Z}/2)$ and $H^*(S^n \setminus M; \mathbb{Z}/2)$ in terms of $H^*(M; \mathbb{Z}/2)$.

5

Let $X = S^{2n} \times S^{2n}$ and define

$$Y = \{(v, w) \in X \mid d(v, w) \leqslant d(v, -w)\}$$
$$Y' = \{(v, w) \in X \mid d(v, w) \geqslant d(v, -w))\}$$

where d is the usual metric on S^{2n} induced by inclusion into $\mathbb{R}^{2n+1}$. Show that Y and Y' are homeomorphic to the unit disk bundle of a vector bundle over S^{2n}. Let $Z = Y \cap Y'$. What is $H^*(Z)$?

Let $Homeo(X, Z)$ be the group of homeomorphisms $f : X \to X$ for which $f(Z) = Z$. Show that $Homeo(X, Z)$ contains a subgroup G isomorphic to D_8 (the dihedral group of order 8) and that the only element of G which is homotopic to 1_X is the identity of G.

How do the elements of G act on $H^*(Z)$? Let $H \subset G$ be the subgroup of those $g \in G$ for which $g(Y) = Y$. How do the elements of H act on $H^*(Y)$ and $H^*(Y, Z)$?

Suppose $n = 1$. For which $g \in G$ is $g|_Z \sim 1_Z$?

END OF PAPER