UNIVERSITY OF CAMBRIDGE

MAMA/113, NST3AS/113, MAAS/113

MAT3

MATHEMATICAL TRIPOS Part III

Monday, 5 June, 2023 1:30 pm to 4:30 pm

PAPER 113

ALGEBRAIC GEOMETRY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define an open immersion of schemes. Give an example of an open immersion $U \to X$ such that X is affine and U is not affine. Give an example of a nontrivial open immersion $V \to Y$ such that Y is connected and affine and V is also affine.

(b) Let R denote the polynomial ring $k[x,y]$ equipped with the standard grading. Let $R^{(2)}$ denote the subring of elements of even degree. Construct an isomorphism between the schemes Proj R and Proj $R^{(2)}$. Construct a closed embedding of Proj $R^{(2)}$ into $\mathbb{P}^2_k$.

(c) Let k be a field. Write $\mathbb{P}^2 = \text{Proj}\ \ k[x,y,z]$ and $\mathbb{A}^1 = \text{Spec}\ k[t]$. Let $Y \subset \mathbb{P}^2 \times_k \mathbb{A}^1$ be the closed subscheme given by the ideal $(xy - tz^2)$.

Let $\pi : Y \to \mathbb{A}^1$ be the morphism induced by the projection. For what points $p \in \mathbb{A}^1$ is the fibre $\pi^{-1}(p)$ an integral scheme? For which points is the fibre reduced?

2

(a) Define properness for a morphism of schemes. Let $\varphi : X \to Y$ be a morphism of schemes. Suppose that $\{U_i\}$ is an open cover of Y such that $\varphi^{-1}(U_i) \to U_i$ is proper for all i. Prove that φ is proper.

(b) Let k be the field of complex numbers. Consider the morphism $\mathbb{A}^1_k \to \mathbb{A}^1_k$ given in coordinates by $t \mapsto t^2$. Let $U \subset \mathbb{A}^1_k$ be the complement of the point $1 \in \mathbb{A}^1_k$. Consider the composite morphism

$$U \subset \mathbb{A}^1 \to \mathbb{A}^1.$$

Prove that the preimage of every closed point is proper, but the morphism is not proper.

(c) What is Cartier divisor? Let D be a Weil divisor on $\mathbb{P}^n$. Show directly from the definition that D is Cartier. It is a fact that the divisor class group of a (noetherian, integral, regular in codimension 1) scheme X coincides with that of $X \times \mathbb{A}^1$. Using this, prove that

$$\text{Cl}(X \times \mathbb{P}^n) = \text{Cl}(X) \oplus \mathbb{Z}.$$

3

(a) What is a quasicoherent sheaf on a scheme? Define the pullback and pushforward operations for quasicoherent sheaves. Let $Z \to X$ be a closed immersion of noetherian schemes. Prove that the pushforward of the structure sheaf $\mathcal{O}_Z$ is coherent. Give an example of a morphism of noetherian schemes $W \to Y$ and a coherent sheaf on W whose pushforward to Y is not coherent.

(b) Let X be a noetherian separated scheme and let $i : Z \to X$ be a closed immersion. Let $\mathcal{F}$ be a coherent sheaf on Z. Using Cech cohomology, prove that

$$H^i(X, i_\star \mathcal{F}) = H^i(Z, \mathcal{F}).$$

Now suppose additionally k is a field and X is $\mathbb{P}^n_k$. Using Cech cohomology prove that $H^i(Z, \mathcal{F})$ vanishes for $i > n$.

(c) Let k be a field and $i : X_d \hookrightarrow \mathbb{P}^n_k$ be a degree d hypersurface. Prove that the kernel of the restriction map

$$\mathcal{O}_{\mathbb{P}^n} \to i_\star \mathcal{O}_{X_d}$$

is isomorphic to the sheaf $\mathcal{O}_{\mathbb{P}^n}(-d)$. If $n = 2$, use this to calculate

$$h^0(X_d, \mathcal{O}_{X_d}) - h^1(X_d, \mathcal{O}_{X_d}).$$

Hence calculate $h^1(X_d, \mathcal{O}_{X_d})$. You may use any fact about the cohomology of line bundles on projective space, provided it is clearly stated.

END OF PAPER