MAT3

MATHEMATICAL TRIPOS Part III

Friday, 2 June, 2023 1:30 pm to 3:30 pm

PAPER 109

COMBINATORICS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Let $A_1, \ldots, A_m$ be odd subsets of $[n]$, with $|A_i \cap A_j|$ even for all $1 \leqslant i < j \leqslant m$. Show that $m \leqslant n$, and that this inequality is best possible for every n.

(ii) Let $A_1, \ldots, A_m$ be even subsets of $[n]$, with $|A_i \cap A_j|$ odd for all $1 \leqslant i < j \leqslant m$. Show that

$$m \leqslant \begin{cases} n-1 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd,} \end{cases}$$

and that these inequalities are best possible for every n.

(iii) Let $p(k)$ be the partition function, the number of partitions of $k \geqslant 1$ into positive integers. Thus $p(1) = 1$, $p(2) = 2$, and $5 = 1 + \cdots + 1 = 1+1+1+2 = 1+1+3 = 1+2+2 = 1+4 = 2+3 = 5$ shows that $p(5) = 7$. Show that for $n = 2k$ there are at least $p(k)$ non-isomorphic families $\{A_1, \ldots, A_n\}$ of odd subsets of $[n]$ such that $|A_i \cap A_j|$ is even for all $1 \leqslant i < j \leqslant n$.

2

(i) Prove the Harris-Kleitman inequality from first principles. [If you quote any related result, such as the Four Functions Theorem, then you must prove that result.]

(ii) Let $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(n)$ be such that if $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then $A \not\subset B$ and $B \not\subset A$. Show that

$$|\mathcal{A}|^{1/2} + |\mathcal{B}|^{1/2} \leqslant 2^{n/2}.$$

[Hint. Let $\mathcal{U}_1 \subset \mathcal{P}(n)$ be the up-set generated by $\mathcal{A}$, i.e. the collection of sets in $\mathcal{P}(n)$ containing at least one member of $\mathcal{A}$, and let $\mathcal{U}_2 = \mathcal{P}(n) \backslash \mathcal{U}_1$. Define $\mathcal{V}_1$ and $\mathcal{V}_2$ analogously for $\mathcal{B}$ instead of $\mathcal{A}$. Check that $\mathcal{A} \subset \mathcal{U}_1 \cap \mathcal{V}_2$ and $\mathcal{B} \subset \mathcal{U}_2 \cap \mathcal{V}_1$.]

3

Let $\mathcal{A}_n = \{A_1, \ldots, A_n\} \subset \mathcal{P}(m)$ be a non-trivial exactly 1-intersecting family of sets, i.e. let $\mathcal{A}_n$ be such that $|A_i \cap A_j| = 1$ for all $1 \leqslant i < j \leqslant n$ and $\cap_{i=1}^n A_i = \emptyset$.

(i) Show that the number of pairs of points covered by the A_i is at least $\binom{n}{2}$, i.e. with $a_i = |A_i|$ we have

$$\sum_{i=1}^{n} \binom{a_i}{2} \geqslant \binom{n}{2},$$

and so $m \geqslant n$.

(ii) For what values of n is there a non-trivial exactly 1-intersecting family $\mathcal{A}_n \subset \mathcal{P}(n)$?

(iii) Are there infinitely many values of n for which there are at least two non-isomorphic non-trivial exactly 1-intersecting families $\mathcal{A}_n \subset \mathcal{P}(n)$?

4

(i) Let $\mathcal{A} \subset \mathcal{P}(X)$ be an antichain, where, as usual, $|X| = n$. For $0 \leqslant h \leqslant n$, write $\mathcal{A}_h = \mathcal{A} \cap X^{(h)}$ for the part of $\mathcal{A}$ at height h. Prove that

$$\sum_{h=0}^{n} |\mathcal{A}_h| \binom{n}{h}^{-1} \leqslant 1, \tag{1}$$

with equality if and only if $\mathcal{A} = \mathcal{A}_h = X^{(h)}$ for some h. Deduce that $|\mathcal{A}| \leqslant \binom{n}{\lfloor n/2 \rfloor}$, with equality if and only if $\mathcal{A} = X^{(\lfloor n/2 \rfloor)}$ or $\mathcal{A} = X^{(\lceil n/2 \rceil)}$.

(ii) Let $r \geqslant 2$, and let $\mathcal{F} \subset X^{(2r)}$ be such that if A, B, C are three sets in $\mathcal{F}$ then $A \cap B \not\subset C$. Show that $|\mathcal{F}| \leqslant \binom{2r}{r} + 1$.

END OF PAPER