MAMA/107, NST3AS/107, MAAS/107

MAT3
MATHEMATICAL TRIPOS Part III

Tuesday, 6 June, 2023 1:30 pm to 4:30 pm

PAPER 107

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let Ω be a domain (i.e. open and connected) in $\mathbb{R}^n$ and consider the equation

$$\Delta u = 0 \tag{1}$$

in Ω.

(a) Suppose that $u \in C^2(\Omega)$.

(i) State and prove the Mean Value Property for u.

(ii) Suppose $x \in \Omega$ and suppose for the moment that u is also $C^\infty(\Omega)$. Show that

$$|\nabla u(x)| \leqslant C \sup_{\Omega} |u|$$

for some constant $C > 0$ independent of u.

(iii) By mollifying u, hence deduce that $u \in C^\infty(\Omega)$.

(b) Suppose next that $u \in H^1(\Omega)$, i.e. u is a *weak* solution of (1). Suppose $0 < \rho' < \rho$, $x \in \Omega$ and $B_\rho(x) \subset \Omega$.

(i) By constructing a suitable test function, prove that weak solutions of (1) satisfy

$$\int_{B_{\rho'}(x)} |\nabla u|^2 \leqslant \frac{1}{|\rho' - \rho|^2} \int_{B_\rho(x)} |u|^2. \tag{2}$$

(ii) Hence deduce that weak solutions of (1) are smooth.

(c) Now suppose that u is merely $L^2_{\text{loc}}(\Omega)$. Formulate what it means for u to satisfy (1). If u satisfies (1) according to your formulation, does u still agree with a smooth function almost everywhere in Ω? Briefly justify your answer.

[You may use without proof the fact that on open, bounded domains $\Omega' \subset \mathbb{R}^n$ with sufficiently smooth boundary, $W^{k,q}(\Omega') \subset C^{r,\alpha}(\Omega')$ for $r + \alpha - k \geqslant n/q$. You may also use without proof standard facts about mollifiers provided they are stated clearly.]

2 Let $\Omega \subset \mathbb{R}^n$ be open and consider the uniformly elliptic differential operator

$$L = a^{ij}\partial_i\partial_j + b^i\partial_i + c,$$

where a^{ij}, b^i, $c \in C^{0,\alpha}(\overline{\Omega})$. Write down what it means for L to be uniformly elliptic.

(a) Consider the case $\Omega = \mathbb{R}^{n-1}_x \times (-1,1)_y$, and $b : \Omega \to \mathbb{R}^n$ given by

$$b(x,y) = (x,y).$$

Suppose that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies

$$Lu \geqslant 0$$

in Ω. Carefully prove the Weak Maximum Principle for u, stating clearly any conditions on c.

(b) Next, consider the case $\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, $a^{ij} = \delta^{ij}$, $b^i = 0$ and $c = 0$, so that $L = \Delta$. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $Lu \geqslant 0$ in Ω. Does u obey the Weak Maximum Principle in this case? Justify your answer with a proof or a counterexample.

(c) Consider now the case $\Omega = \mathbb{R}^n \setminus \overline{B_1(0)}$, $n \geqslant 3$, where $B_1(0)$ is the open unit ball centred at the origin. Let $L = \Delta$. Construct a counterexample to the Weak Maximum Principle in this case. What about the case $n = 2$?

(d) A differential operator $L = \sum_{|\alpha| \leqslant m} a_\alpha(x)\partial^\alpha$ of order m is called elliptic on Ω if

$$\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \neq 0$$

for all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$ with $\xi \neq 0$. Construct an elliptic fourth order differential operator in one dimension on $\Omega = (0,\pi)$ satisfying $a_4(x)a_0(x) < 0$ which does not satisfy the Weak Maximum Principle.

[Hint: for part (c), it may help to consider functions of the form $|x|^a$ in the case $n \geqslant 3$.]

3 Throughout this question, $n \geqslant 2$ and $B_\rho(0)$ denotes the open ball in $\mathbb{R}^n$ with radius ρ and centre the origin.

(a) Let $a^{ij} \in L^\infty(B_1(0))$ for $1 \leqslant i,j \leqslant n$, and suppose that there are constants $\lambda, \Lambda > 0$ such that $a^{ij}(x)\zeta^i\zeta^j \geqslant \lambda|\zeta|^2$ for a.e. $x \in B_1(0)$ and all $\zeta \in \mathbb{R}^n$, and $\sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(B_1(0))} \leqslant \Lambda^2$. Define a differential operator L by $Lu = D_i\left(a^{ij}D_j u\right)$.

(i) Show that there is a constant $C = C(\lambda, \Lambda) > 0$ such that if $u \in W^{1,2}(B_1(0))$ is a non-negative weak subsolution to $Lu = 0$ in $B_1(0)$, then for each constant $\alpha > 1$ and each $\eta \in C_c^1(B_1(0))$,

$$\int_{B_1(0)} |Du|^2 u^{\alpha-2}\eta^2 \leqslant \frac{C}{(\alpha-1)^2}\int_{B_1(0)} u^\alpha |D\eta|^2.$$

Deduce that for each $p > 1$, such u satisfies

$$\sup_{B_{1/2}(0)} u \leqslant C\|u\|_{L^p(B_1(0))}$$

where $C = C(n, \lambda, \Lambda, p) \in (0, \infty)$.

[You may assume without proof the following facts:

- if $u \in W^{1,2}(B_1(0))$ and $u \geqslant \epsilon$ for some $\epsilon > 0$, then for any fixed $\beta > 0$ and any $k \geqslant 1$, the function $v_k = \min\{u^\beta, ku\}$ belongs to $W^{1,2}(B_1(0))$, and its weak derivative is given by $Dv_k = \beta u^{\beta-1}Du$ a.e. on $\Omega_k \equiv \{x \in B_1(0) : u^\beta \leqslant ku\}$ and $Dv_k = kDu$ a.e. on $B_1(0) \setminus \Omega_k$;
- the Sobolev embedding theorem, which says that if $w \in W_0^{1,2}(B_1(0))$ then $w \in L^{2\sigma}(B_1(0))$ and satisfies $\|u\|_{L^{2\sigma}(B_1(0))} \leqslant \overline{C}\|Du\|_{L^2(B_1(0))}$, where $\overline{C} = \overline{C}(n) \in (0,\infty)$, and $\sigma = \frac{n}{n-2}$ if $n \geqslant 3$ and σ is any fixed number > 1, e.g. $\sigma = 2$, if $n = 2$.]

(ii) Carefully stating (without proof), and using in conjunction with (i), any other relevant result proved in the course, deduce that there is a constant $C = C(n, \lambda, \Lambda) \in (0, \infty)$ such that if $u \in W^{1,2}(B_1(0))$ is a non-negative weak solution to $Lu = 0$ in $B_1(0)$ then

$$\sup_{B_{1/4}(0)} u \leqslant C \inf_{B_{1/4}(0)} u.$$

[QUESTION CONTINUES ON THE NEXT PAGE]

(b) Let $v : \mathbb{R}^n \to \mathbb{R}$ ($n \geqslant 2$) be a C^2 function. Suppose that for every ball $B_R(0) \subset \mathbb{R}^n$, v is a minimiser of the area functional $\mathcal{A}(v) = \int_{B_R(0)} \sqrt{1 + |Dv|^2}$ in the sense that $\mathcal{A}(v) \leqslant \mathcal{A}(\widetilde{v})$ for every $\widetilde{v} \in C^2(B_R(0))$ with $\widetilde{v} = v$ on $\partial B_R(0)$.

(i) Show that v satisfies the equation $D_i\left(\frac{D_i v}{\sqrt{1+|Dv|^2}}\right) = 0$ in $\mathbb{R}^n$. Show moreover that for every $R > 0$, the function $v_R(x) = R^{-1}v(Rx)$ also satisfies the same equation in $B_1(0)$.

(ii) Show that for each $k \in \{1, 2, \ldots, n\}$ and each $R > 0$, the partial derivative $w = D_k v_R$ satisfies an equation of the form $D_i\left(a^{ij}(Dv(Rx))D_j w\right) = 0$ weakly in $B_1(0)$, giving an explicit expression for $a^{ij}(p)$, $p \in \mathbb{R}^n$.

(iii) Now suppose that $M \equiv \sup_{\mathbb{R}^n} |Dv| < \infty$. By using the results of (b)(ii) and (a)(ii) above, show in this case that v must be an affine function, i.e. $v(x) = a \cdot x + b$ for some constants $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ and all $x \in \mathbb{R}^n$.

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(a) For a ball $B \subset \mathbb{R}^n$ and a continuous function $\varphi : \overline{B} \to \mathbb{R}$, consider the Dirichlet problem $\Delta v = 0$ in B, $v = \varphi$ on ∂B. Given that a solution $v \in C^\infty(\overline{B})$ to this problem exists whenever $\varphi \in C^\infty(\overline{B})$, deduce that for any $\varphi \in C^0(\overline{B})$ there is a unique solution $v \in C^\infty(B) \cap C^0(\overline{B})$. [You may use without proof standard regularity results for harmonic functions established in the course.]

(b) Let $\Omega \subset \mathbb{R}^n$ ($n \geqslant 2$) be a bounded domain. We say that a function $u \in C^0(\Omega)$ is *sub (super) harmonic* in Ω if for every ball B with $\overline{B} \subset \Omega$, we have that $u \leqslant u_B$ ($u \geqslant u_B$) in $\overline{B}$, where u_B denotes the unique solution in $C^\infty(B) \cap C^0(\overline{B})$ to the Dirichlet problem in (a) above taken with $\varphi = u$.

(i) If $u_1, u_2 \in C^0(\Omega)$ are sub harmonic in Ω, show that $u = \max\{u_1, u_2\}$ is subharmonic in Ω.

(ii) If $u_1, u_2 \in C^0(\overline{\Omega})$, u_1 is sub harmonic in Ω, u_2 is super harmonic in Ω and if $u_1 \leqslant u_2$ on $\partial\Omega$, show that $u_1 \leqslant u_2$ in $\overline{\Omega}$. [Hint: if not, let $M \equiv \sup_{\overline{\Omega}}(u_1 - u_2)$, note that $M > 0$ and first show that the set $\Omega_1 = \{y \in \Omega : (u_1 - u_2)(y) = M\}$ is open].

Now suppose $\varphi \in C^0(\overline{\Omega})$ and let $\mathcal{S}_\varphi = \{u \in C^0(\overline{\Omega}) : u \text{ is sub harmonic in } \Omega \text{ and } u \leqslant \varphi \text{ on } \partial\Omega\}$. Define $\overline{u} : \Omega \to \mathbb{R}$ by

$$\overline{u}(x) = \sup_{u \in \mathcal{S}_\varphi} u(x) \quad \forall x \in \Omega.$$

(iii) Show that $\mathcal{S}_\varphi \neq \emptyset$ and that $\overline{u}$ is well-defined.

(iv) Show that $\overline{u} \in C^\infty(\Omega)$ and that $\Delta \overline{u} = 0$ in Ω. [You may use without proof the following fact: given a subharmonic function $u \in C^0(\Omega)$ and a ball B with $\overline{B} \subset \Omega$, the *harmonic lift* U_B of u with respect to B, defined by $U_B(x) = u_B(x)$ if $x \in B$ and $U_B(x) = u(x)$ if $x \in \Omega \setminus B$, is again subharmonic in Ω.]

(v) Let $z \in \partial\Omega$ and suppose that there is a function $w \in C^0(\overline{\Omega})$ such that w is super harmonic in Ω, $w > 0$ in $\overline{\Omega} \setminus \{z\}$ and $w(z) = 0$. Show that $\overline{u}(x) \to \varphi(z)$ as $x \to z$.

(vi) Let $\mathcal{C}$ denote the cube $[-1, 1]^n \equiv \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -1 \leqslant x_j \leqslant 1 \ \forall j\}$, and let $D = \mathcal{C} \setminus B_{1/4}(p)$ where $p \in B_{1/2}(0)$. Deduce that for any given $\varphi \in C^0(\partial D)$, there is a unique function $u \in C^\infty(D) \cap C^0(\overline{D})$ such that $\Delta u = 0$ in D and $u = \varphi$ on ∂D.

END OF PAPER