UNIVERSITY OF CAMBRIDGE

MAMA/106, NST3AS/106, MAAS/106

MAT3

MATHEMATICAL TRIPOS Part III

Monday, 12 June, 2023 9:00 am to 12:00 pm

PAPER 106

FUNCTIONAL ANALYSIS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove Mazur's theorem.

Let X be a real normed space and (x_n) be a sequence in X. Show that $x_n \xrightarrow{w} 0$ if and only if $f(x_n) \to 0$ for every $f \in X^*$. Show that if $x_n \xrightarrow{w} 0$, then (x_n) is bounded in norm. Show that the following two statements are equivalent:

(i) $x_n \xrightarrow{w} 0$

(ii) For every infinite subset M of $\mathbb{N}$ and for every $\varepsilon > 0$, there exists a finite subset F of M and a convex combination $y = \sum_{n \in F} t_n x_n$ of x_n, $n \in F$, such that $\|y\| < \varepsilon$.

Deduce that for any $p \in (1, \infty)$, if (x_n) is a bounded sequence of pairwise disjointly supported elements of ℓ_p, then $x_n \xrightarrow{w} 0$. [Real sequences $a = (a_i)$ and $b = (b_i)$ are *disjointly supported* if the sets $\{i \in \mathbb{N} : a_i \neq 0\}$ and $\{i \in \mathbb{N} : b_i \neq 0\}$ are disjoint.]

Let A be a commutative unital C^*-algebra. Assume that (a_n) is a bounded sequence in A such that $\varphi(a_n) \to 0$ for every character φ on A. Does it follow that (a_n) converges weakly to 0? Justify your answer.

2

[Throughout this question you may assume results about the spectral theory of Banach algebras.]

State the Holomorphic Functional Calculus.

Let K be a nonempty compact subset of $\mathbb{C}$. Let $\mathcal{R}(\mathcal{K})$ be the closure in $C(K)$ of rational functions without poles in K. Identify, with proof, the character space of $\mathcal{R}(\mathcal{K})$.

State and prove Runge's Approximation Theorem. Deduce that if U is a nonempty open subset of $\mathbb{C}$, then the algebra $\mathcal{R}(\mathcal{U})$ of rational functions without poles in U is dense in the locally convex space $\mathcal{O}(\mathcal{U})$ of holomorphic functions on U.

Let A be a commutative unital Banach algebra. In what follows, given $x \in A$, let $\hat{x}$ denote the Gelfand transform of x, and for a holomorphic function f on an open set containing $\sigma_A(x)$ write $f(x)$ for the image of f under the holomorphic functional calculus for x.

Now let $x_1, x_2 \in A$ be elements such that $\hat{x_1} = \hat{x_2}$. Why is $\sigma_A(x_1) = \sigma_A(x_2)$? Let U be an open subset of $\mathbb{C}$ containing $\sigma_A(x_1)$. Assume that there is an $f \in \mathcal{O}(\mathcal{U})$ such that $f(x_1) = f(x_2)$ and $f'(z) \neq 0$ for all $z \in K$. Show that $x_1 = x_2$. [Hint: Express $f(x_1) - f(x_2)$ as a product $u(x_1 - x_2)$ and show that u is invertible.]

3

[Throughout this question you may assume the theorems of Mazur, Banach–Alaoglu and Goldstine. You may also assume any version of the Hahn–Banach theorems and the fact that $C(K)$ is separable for a compact metric space K.]

Show that a normed space X is separable if and only if the closed unit ball B_{X^*} of X^* is w^*-metrizable.

Prove that if a normed space X is w-separable, then X^* is w^*-separable. Must X^* also be w-separable? Justify your answer.

Show that a Banach space X is reflexive if and only if its closed unit ball B_X is weakly compact.

State the Krein–Milman theorem. Show that if X is a reflexive space, then $B_X = \overline{\text{conv}}\text{Ext}(B_X)$.

Let $f\colon \mathbb{Z} \times \mathbb{Z} \to [0, 1]$ be a function whose value at each point is the average of the values at the four neighbouring points:

$$f(m, n) = \frac{1}{4}\big(f(m + 1, n) + f(m, n + 1) + f(m - 1, n) + f(m, n - 1)\big) \qquad (*)$$

for all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$. Show that f must be constant. [Hint: Note that the set of all such functions is convex. What can you say about extreme points?]

4

(a) State the Hahn–Banach separation theorems in locally convex spaces. [There should be three separate statements depending on the type of objects you are separating.]

State the Banach–Alaoglu theorem and Goldstine's theorem.

Let X be a normed space and Z be a subspace of X^* that separates the points of X: for every $x \in X \setminus \{0\}$ there exists $f \in Z$ with $f(x) \neq 0$. Let σ be the weak topology $\sigma(X, Z)$ on X. Assume that the closed unit ball B_X of X is σ-compact. Prove that X is a dual space: namely that X is isometrically isomorphic to Z^*.

(b) Let H be a real Hilbert space. Let $X = \mathcal{B}(H)$, the Banach space of all bounded linear operators on H. For $x, y \in H$ define $\mu_{x,y}\colon X \to \mathbb{R}$ by $\mu_{x,y}(T) = \langle Tx, y\rangle$. Show that $\mu_{x,y} \in X^*$ with $\|\mu_{x,y}\| = \|x\| \cdot \|y\|$.

Let $M = \{\mu_{x,y} : x, y \in H\}$ and σ be the weak topology $\sigma(X, M)$ on X. Show that the closed unit ball B_X of X is σ-compact. [Hint: For $T \in X$ define the bounded, bilinear form $\theta_T\colon H \times H \to \mathbb{R}$ by $\theta_T(x, y) = \langle Tx, y\rangle$ for all $x, y \in H$, and consider θ_T as an element of a suitable product space. You may assume that for every bounded bilinear from $\varphi\colon H \times H \to \mathbb{R}$, there is a unique operator $T \in X$ such that $\varphi = \theta_T$ and $\|T\| = \|\varphi\|$.] Deduce that $\mathcal{B}(\mathcal{H})$ is a dual space.

5

[Throughout this question you may assume results from the spectral theory of general Banach algebras.]

Let A be a complex algebra. What is a *character* on A? What is the *character space* Φ_A of A? Show that a character φ on a unital Banach algebra A is bounded with $\|\varphi\|= 1$.

Let A be a commutative unital Banach algebra. Prove that if $x \in A$, then $\sigma_A(x) = \{\varphi(x) : \varphi \in \Phi_A\}$. Deduce that $\Phi_A \neq \emptyset$.

Let Y and Z be isomorphic Banach spaces of non-zero dimension. Let $X = Y \oplus Z$ with norm $\|(y, z)\| = \|y\| + \|z\|$ for all $y \in Y$, $z \in Z$. Show that $\Phi_{\mathcal{B}(X)} = \emptyset$. [Hint: Consider suitable idempotents in $\mathcal{B}(X)$.]

Let A be a unital C^*-algebra. Show that if $x \in A$ is hermitian, then $\sigma_A(x) \subset \mathbb{R}$. Show further that if B is a unital C^*-subalgebra of A and $x \in B$ is normal, then $\sigma_B(x) = \sigma_A(x)$.

State the Commutative Gelfand–Naimark theorem. Show that a positive element in a unital C^*-algebra has a positive square root.

Let A be a unital C^*-algebra. Prove the following statements:

1. For every $x \in A$ there exist unique hermitian elements $h, k \in A$ such that $x = h+ik$.
2. If $x \in A$ is hermitian with $\|x\| \leqslant 1$, then $1 - x^2$ is positive. Hence $x = \frac{1}{2}(u + u^*)$ for some unitary element $u \in A$.
3. If $x \in A$ is hermitian with $\|t1 - x\| \leqslant t$ for some $t \in \mathbb{R}$, then x is positive.
4. If $x \in A$ is positive with $\|x\| \leqslant t$ for some $t \in \mathbb{R}$, then $\|t1 - x\| \leqslant t$.
5. If $x, y \in A$ are positive, then $x + y$ is also positive.

END OF PAPER