MAMA/105, NST3AS/105, MAAS/105

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 7 June, 2023 9:00 am to 12:00 pm

PAPER 105

ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) State the Cauchy–Kovalevskaya Theorem, restricted to quasilinear second order scalar equations on $\mathbb{R}^n$. Give in particular the definition of what it means for a hypersurface Γ to be non-characteristic at x_0 with respect to initial data (u_0, u_1).

(b) Indicate which of these hypersurfaces in $\mathbb{R}^{n+1}$ (with coordinates t, $x_1,\ldots, x_n$) are non-characteristic, briefly justifying your answer. Remember that the answer may depend on the point and the data.

(i) the hypersurface $t = 0$ for the equation $\Box u := -\partial_t^2 + \Delta u = 0$

(ii) the hypersurface $t = 0$ for the equation $\partial_t^2 u + \Delta u = 0$

(iii) the hypersurface $t = 0$ for the equation $\partial_t u = \Delta u$

(iv) the hypersurface $t = 0$ for the equation $\partial_t^2 u = (1 + u^2)\Delta u$

(v) the hypersurface $x_1^2 + \cdots + x_n^2 = 1$ for the equation $\Box u = 0$

(vi) the hypersurface $t = x_1$ for the equation $\Box u = 0$

(vii) the hypersurface $t = x_1$ for the equation $\partial_t u = \Delta u$.

(c) Show that that there exists a unique local analytic solution in a neighbourhood of the origin $(0, 0)$ to the following characteristic initial value problem in $\mathbb{R}^{1+1}$ with coordinates (t, x):

$$\Box u = u, \qquad u|_\Gamma = u_0 \tag{1}$$

where $\Gamma = \{t - x = 0, 0 \leqslant t + x \leqslant 1\} \cup \{t + x = 0, 0 \leqslant t - x \leqslant 1\}$, and u_0 is the restriction to Γ of an analytic function defined on some open neighbourhood of Γ.

[Hint: Rewrite the equation in terms of $\eta = t + x$ and $\xi = t - x$. Compute the power series at $(0, 0)$ from the equation and the data and estimate the growth of the terms to show the existence of a solution.]

(d) Show moreover that an analytic solution in fact exists in the characteristic square:

$$\{0 \leqslant t - x \leqslant 1\} \cap \{0 \leqslant t + x \leqslant 1\}. \tag{2}$$

[Hint: Look at (ξ_0, η_0) with $\xi_0 \leqslant 1$, $\eta_0 \leqslant 1$ such that there exists an analytic solution on $\{0 \leqslant \xi \leqslant \xi_0\} \cap \{0 \leqslant \eta \leqslant \eta_0\} \setminus \{(\xi_0, \eta_0)\}$ and estimate the solution appropriately in this region by integrating along characteristics. Show from this that the solution extends analytically to (ξ_0, η_0) and infer the result.]

(e) Show that there exists a C^2 solution u to the initial value problem (1) in the region (2), where u_0 is now only assumed to be the restriction to Γ of a C^2 function defined on some neighbourhood of Γ. [Hint: Approximate by analytic u_0 and use the estimates proven in (d).]

2 Consider a domain $\Omega \subset \mathbb{R}^n$ with compact closure and smooth boundary $\partial\Omega$.

(a) Define the space $H_0^1(\Omega)$ together with its inner product, showing that the inner product is positive definite. [You may use without proof the Poincaré inequality.]

Given $f \in L^2(\Omega)$, state what it means for a $u \in H_0^1(\Omega)$ to be a *weak solution* of the problem

$$\Delta u = f, \qquad u|_{\partial\Omega} = 0. \tag{1}$$

(b) For $f \in L^2(\Omega)$, show the existence of a unique weak solution u of (1). [You may use without proof the Riesz representation theorem.]

(c) Consider now the Dirichlet problem:

$$\Delta u - u = f, \qquad u|_{\partial\Omega} = 0. \tag{2}$$

Formulate a notion of weak solution for (2) for $f \in L^2(\Omega)$ and prove again the existence of a unique weak solution. [Hint: Show that there is a positive definite inner product intimately related to (2).]

(d) State (without proof) a combined boundary and interior regularity for (1), i.e. an estimate for an appropriate high Sobolev norm of u in terms of an appropriate slightly lower Sobolev norm of f. Use this to prove an analogous interior and boundary regularity for (2).

(e) Consider the Dirichlet problem for the coupled *nonlinear* system

$$\Delta u = (\partial_{x_1}^2 v)^2 + \varepsilon f, \qquad \Delta v + v = (\partial_{x_2}^2 u)^2, \qquad u|_{\partial\Omega} = 0 = v|_{\partial\Omega}, \tag{3}$$

where $x_1, x_2, \ldots, x_n$ denote the coordinates of $\mathbb{R}^n$. Using (c) and (d), show that given $f \in C^\infty(\overline{\Omega})$, there exists an $\varepsilon_0 > 0$ sufficiently small such that for all $0 \leqslant \varepsilon < \varepsilon_0$, there exists a smooth solution $(u, v) \in C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})$ of the problem (3). [You may use without proof the fact, shown on example sheets, that $H^k(\Omega)$ is an algebra for sufficiently high k and that $\|gh\|_{H^k(\Omega)} \leqslant C_k(\Omega)\|g\|_{H^k(\Omega)}\|h\|_{H^k(\Omega)}$.]

3 Let u_0, u_1 be smooth functions on $\mathbb{R}^3$ of compact support.

(a) Show the existence of a C^∞ smooth solution $u : \mathbb{R}^{3+1} \to \mathbb{R}$ to

$$\Box u = 0, \qquad u(0,x) = u_0(x), \qquad \partial_t u(0,x) = u_1(x). \tag{1}$$

Show that any other C^2 function $\tilde{u} : \mathbb{R}^{3+1} \to \mathbb{R}$ satisfying (1) coincides with u. Is the assumption of compact support on u_0 and u_1 necessary for these statements?

(b) Show that the solution u satisfies the *strong* version of Huygens' principle, which you should formulate.

(c) Let (r, θ, ϕ) denote standard spherical coordinates on $\mathbb{R}^3$. Define $\xi = t - r$ and $\eta = t + r$, consider ru as a function of $(\xi, \eta, \theta, \phi)$, and define

$$\psi_+(\xi, \theta, \phi) = \lim_{\eta \to \infty} ru(\xi, \eta, \theta, \phi)$$

and

$$\psi_-(\eta, \theta, \phi) = \lim_{\xi \to -\infty} ru(\xi, \eta, \theta, \phi).$$

Show that $\psi_\pm$ are well defined functions $\psi_\pm : \mathbb{R} \times \mathbb{S}^2 \to \mathbb{R}$ of compact support.

(d) Restrict now to the case where (u_0, u_1) are spherically symmetric and can thus be considered as radial functions $u_0(r)$, $u_1(r)$ in the space

$$Z = \{(u_0(r), u_1(r)) \in C_c^\infty([0,\infty)) \times C_c^\infty([0,\infty)) : \partial^n u_0(0) = 0 = \partial^n u_1(0), n \text{ odd}\}.$$

Show that for for $(u_0, u_1) \in Z$, the functions $\psi_\pm$ are smooth, spherically symmetric and of compact support, and thus can be viewed as functions $\psi_\pm(r) \in C_c^\infty(\mathbb{R})$, and moreover, that the associations

$$(u_0, u_1) \mapsto \psi_+, \qquad (u_0, u_1) \mapsto \psi_- \tag{2}$$

both define injective linear maps

$$F_\pm : Z \to C_c^\infty(\mathbb{R}).$$

Identify explicitly the images of these maps $X_\pm \subset C_c^\infty(\mathbb{R})$ and compute moreover explicitly the resulting "scattering" map $S := F_+ \circ F_-^{-1}$

$$S : X_- \to X_+$$

taking $\psi_- \mapsto \psi_+$.

END OF PAPER