UNIVERSITY OF CAMBRIDGE

MAMA/102, NST3AS/102, MAAS/102

MAT3

MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2023 9:00 am to 12:00 pm

PAPER 102

LIE ALGEBRAS AND THEIR REPRESENTATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **Question 3** and **THREE** other questions.
There are **FIVE** questions in total.
The questions carry equal weight.

All Lie algebras are over $\mathbb{C}$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Let G be an algebraic group. Define the Lie algebra $\mathfrak{g}$ of G, and show that the Lie bracket is antisymmetric and that the Jacobi identity holds.

[If you wish, you may use without proof the identity that for all $x, y, z \in G$,

$$(x, y^{-1}, z)^y.(y, z^{-1}, x)^z.(z, x^{-1}, y)^x = 1$$

where $(x, y) = x^{-1}y^{-1}xy$, $(x, y, z) = ((x, y), z)$, and $a^b = b^{-1}ab$.]

(ii) Give an example of two distinct groups G with Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$.

(iii) For each of those groups G, describe the irreducible representations of G. [You do not have to prove your answer is correct.]

(iv) If $\mathfrak{g}$ is a Lie algebra, define what it means for a bilinear form $(,) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ to be invariant. Let $(,)$ be a non-degenerate invariant symmetric bilinear form. Define the Casimir element Ω and show that $\Omega x = x\Omega$ for all $x \in \mathfrak{g}$.

(v) Let $\mathfrak{g} = \mathfrak{sl}_2$. Write Ω explicitly in terms of the standard basis of $\mathfrak{g}$. If L is an irreducible module for $\mathfrak{g}$ of dimension $n + 1$, then Ω acts on L by multiplication by some $\lambda \in \mathbb{C}$. Compute λ.

(vi) Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{s}$ a subalgebra. If L is an irreducible representation of $\mathfrak{g}$, must L be completely reducible as a representation of $\mathfrak{s}$? [You must justify your answer.]

2

(i) State Cartan's criteria for a Lie algebra over $\mathbb{C}$ to be solvable.

(ii) Let $\mathfrak{t} \subseteq \mathfrak{g}$, $\mathfrak{g}$ a Lie algebra. Define what it means for $\mathfrak{t}$ to be a *torus*, and define the weight space decomposition of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$, $\mathfrak{t} \subseteq \mathfrak{g}$ a maximal torus, and suppose $\mathfrak{g} = \mathfrak{g}_\alpha \oplus \mathfrak{t} \oplus \mathfrak{g}_\beta$ is the weight space decomposition.

Further suppose that $\mathfrak{g}$ has a non-degenerate trace form $(,)_V$.

Prove that $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{h}$, where $\mathfrak{h} \subseteq \mathfrak{t}$ is an abelian Lie algebra of dimension $\dim \mathfrak{t} - 1$ and the direct sum is as Lie algebras— $\mathfrak{h}$ and $\mathfrak{sl}_2$ commute.

(iii) Find a representation V of $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{h}$, $\mathfrak{h}$ abelian, such that the induced trace form is non-degenerate.

3

Let

$$\mathfrak{g} = \mathfrak{so}_{2n} = \left\{ A \in Mat_{2n} | AJ + JA^T = 0 \right\},$$

where $J = \begin{pmatrix} & & & & & 1 \\ & & & & \cdot^{\cdot^{\cdot}} & \\ & & & 1 & & \\ & & 1 & & & \\ & \cdot^{\cdot^{\cdot}} & & & & \\ 1 & & & & & \end{pmatrix}$, and let $\mathfrak{t} =$ diagonal matrices in $\mathfrak{g}$.

(i) Decompose $\mathfrak{g}$ as a $\mathfrak{t}$-module, and hence write the roots R for $\mathfrak{g}$. Choose positive roots to be those occurring in upper triangular matrices. Write down the positive roots R^+, the simple roots Π, the highest root θ, and the fundamental weights. Write down ρ.

Write the root lattice P, the weight lattice Q, and determine Q/P.

Draw the Dynkin diagram and label it by simple roots. Draw the extended Dynkin diagram.

(ii) Show that $\mathfrak{so}_6 \simeq \mathfrak{sl}_4$.

(iii) For each root $\alpha \in R$, write the reflection $s_\alpha : \mathfrak{t} \to \mathfrak{t}$ explicitly. Describe the Weyl group W (you do not need to prove your answer).

4

(i) State the Weyl character formula and the Weyl dimension formula, briefly defining the notation you use.

(ii) Draw the root system of G_2, and the fundamental weights ω_1, ω_2.

Let ω_1 denote the shorter fundamental weight.

Show that the representation with highest weight ω_2 is the adjoint representation.

Draw the weights of the representation with highest weight ω_1, and its crystal.

Write down the dimension of the irreducible representation with highest weight $n_1\omega_1 + n_2\omega_2, \quad n_1, n_2 \in \mathbb{N}$.

5

Let $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots of $\mathfrak{g}$, with α_n the *short* simple root. Let ω_n be the fundamental weight dual to $\alpha_n^\vee$.

Using the Littelmann path model, or otherwise, show that the crystal $\mathbb{S} = B(\omega_n)$ of the representation $L(\omega_n)$ of $\mathfrak{so}_{2n+1}$ is as follows: $B = \{(i_1, \ldots, i_n) \mid i_j = \pm 1\}$ with $wt((i_1, \ldots, i_n)) = \frac{1}{2}\sum i_j e_j \in P$, and

$$\widetilde{e_j}(i_1, \ldots, i_n) = \begin{cases} (i_1, \ldots, \underset{j}{+1}, \underset{j+1}{-1}, \ldots, i_n) & \text{if } (i_j, i_{j+1}) = (-1, +1) \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leqslant j \leqslant n-1$, and

$$\widetilde{e_n}(i_1, \ldots, i_n) = \begin{cases} (i_1, \ldots, i_{n-1}, 1) & \text{if } i_n = -1 \\ 0 & \text{otherwise.} \end{cases}$$

When $n = 3$ draw the crystal $\mathbb{S}$ and decompose $\mathbb{S} \otimes \mathbb{S}$ into irreducibles.

END OF PAPER