MAMA/101, NST3AS/101, MAAS/101

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday, 7 June, 2023 9:00 am to 12:00 pm

PAPER 101

COMMUTATIVE ALGEBRA

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

Unless stated otherwise, you may use, without proving, every claim proved in the lectures and every claim from the examples sheets, as long as you state the claim accurately.

The term *ring* stands for a commutative unital ring, and the term *module* stands for a module over such a ring.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

a. For each of the following statements, prove or give a counter-example:

(i) Every noetherian ring is artinian.

(ii) Every artinian ring is noetherian.

(iii) Every noetherian module is artinian.

(iv) Every artinian module is noetherian.

b. Let $(A, \mathfrak{m})$ be an artinian local ring such that $|A/\mathfrak{m}| < \infty$. Prove that $|A| < \infty$.

c. Let $f\colon B^m \to B^n$ be an injective B-module homomorphism, where B is a nonzero ring. Prove that $m \leqslant n$.

[**Hint:** It may be helpful to consider a homomorphism of the form $(x_1, \ldots, x_p) \mapsto (x_1, \ldots, x_p, \underbrace{0, \ldots, 0}_{q \text{ times}})$ for some integers p and q.]

2

a. What does it mean for a ring extension to be: (i) integral, (ii) finite?

b. Let $A \subset B$ be rings and $\mathfrak{p} \in \operatorname{spec} A$, $S = A \setminus \mathfrak{p}$, $B_{\mathfrak{p}} = S^{-1}B$. Consider the ring homomorphism $B \to B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ given by composing the localization map $B \to B_{\mathfrak{p}}$ and the quotient map $B_{\mathfrak{p}} \to B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$. Give a simple description of the image of the contraction map $\operatorname{spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) \to \operatorname{spec} B$.

c. Consider the statement: "for a ring extension $A \subset B$ and $\mathfrak{p} \in \operatorname{spec} A$, the set of prime ideals of B lying over $\mathfrak{p}$ is finite."
Is the statement true in general? Is it true assuming $A \subset B$ is integral? Is it true assuming $A \subset B$ is finite?

d. Let $A \subset B$ be a finite extension of rings, let k be an algebraically closed field, and let $f\colon A \to k$ be a ring homomorphism. Write H_f for the set of ring homomorphisms $g\colon B \to k$ such that $g(a) = f(a)$ for all $a \in A$. Prove that H_f is finite but not empty. [**Hint:** You may use, without proof, that the statement is true when A and B are fields.]

3

a. State Zariski's Lemma and the Strong Nullstellensatz.

Let $A = k[T_1, \ldots, T_n]$ be the ring of polynomials in $T_1, \ldots, T_n$ over an algebraically closed field k. Let $\mathfrak{a}$ be an ideal of A such that $|V(\mathfrak{a})| = 1$, where $V(\mathfrak{a}) = \{(x_1, \ldots, x_n) \in k^n \mid f(x_1, \ldots, x_n) = 0 \; \forall f \in \mathfrak{a}\}$. Prove or disprove: there is a maximal ideal $\mathfrak{m}$ of A, and $r \geqslant 1$, such that $\mathfrak{m}^r \subset \mathfrak{a} \subset \mathfrak{m}$.

b. Fix $n \geqslant 0$. For a field k and a subset S of the polynomial ring $k[T_1, \ldots, T_n]$, let $V_k(S) = \{(x_1, \ldots, x_n) \in k^n \mid f(x_1, \ldots, x_n) = 0 \; \forall f \in S\}$.
Let I and J be ideals of $\mathbb{Z}[T_1, \ldots, T_n]$ such that $V_{\mathbb{C}}(I) \subset V_{\mathbb{C}}(J)$.
For a prime number p, consider the map $\pi_p \colon \mathbb{Z}[T_1, \ldots, T_n] \to \mathbb{F}_p[T_1, \ldots, T_n]$ such that $\pi_p(f)$ is obtained from f by applying the quotient map $\mathbb{Z} \to \mathbb{F}_p$ to each cofficient of f. Prove that $V_{\overline{\mathbb{F}}_p}(\pi_p(I)) \subset V_{\overline{\mathbb{F}}_p}(\pi_p(J))$ for all but finitely many prime numbers p (here $\overline{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$).

c. What does it mean for a module to be flat?

Let M be a module over a PID A. Prove that M is flat if and only if M is torsion free. [you are required to prove both directions, even though one of them was shown in class.]

4

Recall: For an integral domain A, if every irreducible element of A is prime and every noninvertible nonzero element of A is a finite product of irreducible elements of A then A is a UFD.

a. Let $\mathcal{F}$ be a family of proper ideals of a ring A such that for every ideal I of A and $a \in A$, if $(I : a) \notin \mathcal{F}$ and $I + (a) \notin \mathcal{F}$ then $I \notin \mathcal{F}$. Let J be a maximal element of $\mathcal{F}$ (with respect to inclusion). Prove that J is prime.

[Reminder: $(I : a) = \{x \in A \mid xa \in I\}$]

b. Prove that if all prime ideals of an integral domain A are principal then A is a PID.

c. Let A be a noetherian integral domain. Prove that A is a UFD if and only if every minimal nonzero prime ideal of A is principal

d. Let A be a ring. Prove that the following are equivalent:

(i) A is a PID and not a field.

(ii) A is a noetherian UFD of Krull dimension 1.

[You may use, without proof, that every PID is a UFD.]

5

a. Let $A \subset B \subset C$ be rings, where (i) A is noetherian , (ii) C is finitely generated as an A-algebra, and (iii) C is finite over B. Prove that B is finitely generated as an A-algebra.

b. Prove that every infinite field is not finitely generated as an algebra over the integers.

c. Let $A = \bigoplus_{n \geqslant 0} A_n$ and $B = \bigoplus_{n \geqslant 0} B_n$ be noetherian graded algebras over a field $k = A_0 = B_0$. Make $A \otimes_k B$ into a graded k-algebra by letting $a \otimes b$ be homogeneous of degree $i + j$ whenever a (resp. b) is homogeneous of degree i (resp. j).

Write P_A, P_B and $P_{A \otimes B}$ for the Poincaré series of A, B and $A \otimes_k B$, respectively (here we are considering Poincaré series and Hilbert polynomials with respect to the additive function $V \mapsto \dim_k V$.)

(i) What is the definition of the Poincaré series of A? (i.e. what is P_A?)

(ii) What is the definition of the Hilbert polynomial of A (when it exists)?

(iii) Give an example of A as above that does not have a Hilbert polynomial.

(iv) State the sufficient condition, proved in the lectures, for A to have a Hilbert polynomial, and express the degree of the Hilbert polynomial of A in terms of P_A.

(v) Express $P_{A \otimes B}$ in terms of P_A and P_B.

END OF PAPER