UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 10 June, 2022 1:30 pm to 3:30 pm

PAPER 354

GAUGE/GRAVITY DUALITY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **BOTH** questions.
There are **TWO** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Derive the unitary bound for an *antisymmetric* tensor primary operator $\mathcal{O}^{ij}$ in a CFT of general dimension $d \geqslant 4$. Show that there is a more restrictive unitary bound when $d = 3$.

[*Hint: in Euclidean signature, the nonzero commutators in the conformal algebra are:*

$$\begin{aligned} [D, P_a] &= iP_a, & [D, K_a] &= -iK_a, \\ [P_a, M_{bc}] &= i(\delta_{ab}P_c - \delta_{ac}P_b), & [K_a, M_{bc}] &= i(\delta_{ab}K_c - \delta_{ac}K_b), \\ [K_a, P_b] &= 2i(\delta_{ab}D - M_{ab}), \\ [M_{ad}, M_{bc}] &= i(\delta_{ab}M_{cd} - \delta_{ac}M_{bd} - \delta_{bd}M_{ca} + \delta_{cd}M_{ba}). \end{aligned}$$

with D, M_{ab}, P_a and K_a being the generators of dilations, rotations, translations, and special conformal transformations respectively.]

(b) A noninteracting massless 2-form potential in AdS$_6$ is described by the following action:

$$I = \frac{1}{12}\int d^6\mathbf{x}\sqrt{-g}F_{abc}F^{abc},$$

where $F_{abc} = A_{ab,c} + A_{bc,a} + A_{ca,b}$, the potential is antisymmetric: $A_{ab} = -A_{ba}$, and there is a gauge-symmetry $A_{ab} \to A_{ab} + \alpha_{a,b} - \alpha_{b,a}$ (for an arbitrary 1-form α_a). Suppose that this field arises in the bulk dual to a large N holographic CFT$_5$. (Assume we are in a weak coupling limit where we can neglect gravity and other interactions.) Write down the wave equation for A_{ab} in the AdS-Poincaré spacetime:

$$ds^2 = \frac{1}{z^2}\left(dz^2 + \eta_{ij}\,dx^i dx^j\right),$$

where $i, j = 0 \ldots 4$. Determine the behavior of the field as $z \to 0$, and thereby calculate the dimensions Δ of the corresponding (single-trace, primary) *source* J_{ij} and *operator* $\mathcal{O}^{ij}$ in the CFT under dilations. Explain how you know which one is the source, and which one is the operator.

(c) In a CFT$_d$ in Minkowski spacetime, the momentum-space stress-tensor is:

$$T_{ij}(\mathbf{p}) = \frac{1}{(2\pi)^{d/2}}\int d^4\mathbf{x}\, e^{i\mathbf{p}\cdot\mathbf{x}}T_{ij}(\mathbf{x}).$$

Using symmetry and other physical principles, show that (in $d > 2$, and for a certain range of momenta $\mathbf{p}$ and $\mathbf{q}$) its 2 point function takes the form:

$$\langle 0|\, T_{ij}(\mathbf{q})T_{kl}(\mathbf{p})\,|0\rangle \propto \delta^d(\mathbf{p}-\mathbf{q})|\mathbf{p}|^\beta \left(P_{ik}P_{jl} + P_{il}P_{jk} - \frac{2}{d}P_{ij}P_{kl}\right),$$

where $P_{ij} = \eta_{ij} + \frac{p_i p_j}{p^2}$. (Do not concern yourself with the overall multiplicative factor of the 2 point function, which depends on the particular theory.) Determine the value of the power β. Comment on the range of momenta for which this formula is valid, and what happens outside of this range.

Assuming the CFT is holographic, give a bulk interpretation of the intermediate states $T_{ij}(\mathbf{p})\,|0\rangle$.

2

The metric of AdS_3 is:

$$ds^2 = \frac{R_{\mathrm{AdS}}^2}{(\cos r)^2}\left(-dt^2 + dr^2 + (\sin r)^2\, d\phi^2\right),$$

where $0 \leqslant r < \pi/2$, ϕ is periodic with $0 \leqslant \phi < 2\pi$, and R_{AdS} is the AdS radius.

(a) The entanglement entropy on any interval in a CFT_2 is given by

$$\frac{c}{3}\ln(\epsilon^{-1}) + \text{finite}.$$

where ϵ is a UV cutoff. Confirm this formula using the holographic entropy formula. You may select any choice of boundary interval. Note that the CFT_2 has central charge $c = 3R_{\mathrm{AdS}}/2G$ in units where $\hbar = 1$.

[*Hint: The following Laurent series may be useful:* $\dfrac{1}{\sin x} = \csc x = \dfrac{1}{x} + \dfrac{x}{6} + \dfrac{7x^3}{360} + \ldots$]

(b) (i) Evaluate the action of general relativity (with Gibbons-Hawking term)

$$I_{GR} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^3\mathbf{x}\sqrt{-g}\,(R - 2\Lambda) \;-\; \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2\mathbf{y}\sqrt{-h}\,K$$

on the submanifold $\mathcal{M} \subset \mathrm{AdS}_3$ consisting of points with $r \leqslant (\pi/2) - \epsilon$. Here h_{ij} is the pullback of the metric to the boundary $\partial\mathcal{M}$, whose extrinsic curvature is $K := \frac{1}{2}n^a g_{ij,a} g^{ij}$ (n^a being the inward-pointing normal vector). (Since the manifold is time-translation invariant, there will be an overall infinite factor of Δt in your answer.)

(ii) Show that the divergent contribution to I can be eliminated by adding a local counterterm $I_{c.t.}$ to the boundary. (Do not consider the factor of Δt to be a divergence, as with respect to the CFT this is an IR divergence rather than a UV one.) Calculate this counterterm for the boundary ∂M explicitly, in the limit where $\epsilon \to 0$.

(iii) The remaining finite part (which you are not asked to calculate) can be written in the form:

$$\frac{1}{\Delta t}\lim_{\epsilon\to 0}(I + I_{c.t.}) = -\frac{c}{12},$$

where c is the central charge of the CFT_2. What is the physical significance of the right hand side of this equation, from the perspective of the boundary theory?

END OF PAPER