UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2022 9:00 am to 11:00 am

PAPER 347

ASTROPHYSICAL BLACK HOLES

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Consider a black hole with mass $M_{\rm BH}$ and spin $s_{\rm BH}$, shining at the fraction f of the Eddington luminosity, $L_{\rm Edd}$. Derive the expression for the Eddington luminosity knowing that the gas is characterised by some typical opacity κ. Then taking advantage of the force multiplier, $M(r) = \frac{a_{\rm rad}(r)}{a_{\rm Compton}(r)}$, where $a_{rad}(r)$ is the total acceleration due to radiation pressure force at distance r and $a_{\rm Compton}(r)$ is the radiation pressure acceleration in the Compton limit, write down the appropriate momentum equation for the spherically symmetric outflow. Discuss in which cases this outflow can exist based on the possible range of $M(r)$ values.

Estimate now the black body temperature of this source, $T_{\rm BB}$, close to the innermost stable circular orbit assuming that the black body radiation is a good description of the radiated power. How does $T_{\rm BB}$ depend on the black hole mass and spin? Outline the physical meaning of these dependences.

(b) A supermassive black hole accretes at a constant rate such that at $z = 6$ its mass is $\sim 5 \cdot 10^8\,{\rm M}_\odot$, where ${\rm M}_\odot$ is the solar mass. Assuming radiative efficiency $\epsilon = 0.1$, estimate the possible range of lifetimes of this quasar. Is it possible to put relevant constraints on the seed mass and the seeding redshift for this black hole growth to happen? You may consider either sub- or super-Eddington accretion.

Another supermassive black hole with the same mass at $z = 6$ accumulated its mass through Eddington-limited accretion. Assuming again $\epsilon = 0.1$ estimate the possible range of quasar lifetimes of this source and compare it to the previous estimates.

Hence deduce what next-generation high redshift observations of quasars probing down to sufficiently low luminosities can tell us about the assembly of supermassive black holes.

Which two arguments could you use to constrain the stellar mass of the host galaxy of these quasars and how large you expect it to be? What does the presence of high redshift quasars tells us about the stellar mass assembly of the host galaxies?

[*Recall* $z = 6$ *corresponds to* ~ 1 Gyr*;* $M_\odot \sim 2 \cdot 10^{33}$ g*;* $c \sim 3 \cdot 10^{10}\,{\rm cm\,s^{-1}}$*, where* c *is the speed of light;* $L_{\rm Edd} \sim 1.3 \cdot 10^{38} \frac{M_{\rm BH}}{M_\odot}\,{\rm erg\,s^{-1}}$ *in the electron scattering regime;* $t_{\rm Edd} = \frac{c\sigma_{\rm T}}{4\pi G m_{\rm p}} \sim 0.45$ Gyr*, where* $\sigma_{\rm T}$ *is the Thomson cross-section,* G *is the gravitational constant and* $m_{\rm p}$ *is the proton mass;* $\exp(20) \sim 5 \cdot 10^8$ *and* $\exp(6.3) \sim 5 \cdot 10^2$.]

2 (a) Detail the ten assumptions inherent to the standard Shakura-Sunyaev accretion disc. Hence, write down the expression for the mass accretion rate through this disc, $\dot{m}$, starting from the generic mass conservation equation.

(b) Recall that the ϕ component of the Navier-Stokes equation can be expressed as

$$\Sigma\left(u_{\mathrm{R}}\frac{\partial u_\phi}{\partial R}+\frac{u_{\mathrm{R}}u_\phi}{R}\right)=\frac{1}{R^2}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\partial(u_\phi/R)}{\partial R}\right), \tag{1}$$

where R is the radius in cylindrical-polar coordinates, u_{R} and u_ϕ are radial and azimuthal velocities, Σ is the surface density of the accretion disc, and ν is the kinematic viscosity. With the aid of this equation, or otherwise, derive an expression for the u_{R} which depends on ν, Σ and R only.

(c) Consider now an astrophysical system where the matter is injected at a constant rate $\dot{m}_0$ into Keplerian orbits at radius R_0 around the central supermassive black hole. Within R_0 the system behaves as the standard Shakura-Sunyaev disc. Outside R_0 there is a steady distribution of mass that extends to very large radii and removes the angular momentum of the matter accreting within R_0.

Show that in a steady state

$$\nu\Sigma=\frac{\dot{m}_0}{3\pi}\left[1-\left(\frac{R_{\mathrm{ISCO}}}{R}\right)^{\frac{1}{2}}\right], \tag{2}$$

for $R<R_0$, where R_{ISCO} is the radius of the innermost stable circular orbit, and

$$\nu\Sigma=\frac{\dot{m}_0}{3\pi}\left[\left(\frac{R_0}{R}\right)^{\frac{1}{2}}-\left(\frac{R_{\mathrm{ISCO}}}{R}\right)^{\frac{1}{2}}\right], \tag{3}$$

for $R>R_0$.

[*Hint: Consider carefully boundary conditions at* R_{ISCO} *and* R_0.]

(d) Viscous dissipation rate of a Shakura-Sunyaev accretion disc, F_{diss}, is given by

$$F_{\mathrm{diss}}=\nu\Sigma R^2\left(\frac{\partial\Omega}{\partial R}\right)^2, \tag{4}$$

where Ω is the angular velocity. Derive an expression for the radial dependence of the effective temperature, T_{eff}, of the disc under the black body assumption.

Find the radius where T_{eff} is maximal and determine how T_{eff} scales with radius at large distances.

Now calculate the total luminosity emitted by the disc and comment how it compares with the rate of gravitational energy loss due to inflow.

Compare the emitted luminosity outside some radius R with the gravitational energy released at radius R. What can you deduce from this comparison?

3 (a) Consider a quasar generating a spherically symmetric Eddington wind which is radiation-pressure-driven with gas optical depth due to electron scattering, $\tau_{\rm e.s.} \approx 1$. Sketch the different spatial regions that exist starting from the quasar radially outwards to the unperturbed interstellar medium.

Describe the physical reasons that may lead either to an energy-driven or momentum driven outflow and write down the appropriate relation for the rate of change of momentum of the shocked, swept-up shell caused by the wind. Explain the physical reason for the possible momentum-boost of the shell in the energy-driven limit.

(b) A radiation-pressure-driven outflow is generated by a quasar with luminosity L. If the optical depths in the UV and IR are, $\tau_{\rm UV}$ and $\tau_{\rm IR}$, respectively, write down the general expression for the critical luminosity, $L_{\rm crit}$, where the radiation pressure balances gravity.

Assuming now that the host galaxy is well described by the singular isothermal sphere with a constant velocity dispersion σ and gas fraction $f_{\rm gas}$, show that the critical luminosity in the IR, single-scattering and optically thin UV limit is

$$L_{\rm crit,IR} = \frac{8\pi c \sigma^2 r}{\kappa_{\rm IR}}\,, \quad L_{\rm crit,s.s.} = \frac{4 c f_{\rm gas} \sigma^4}{G}\,, \quad L_{\rm crit,UV} = \frac{8\pi c \sigma^2 r}{\kappa_{\rm UV}}\,, \tag{1}$$

respectively, where r is the radial distance from the quasar, c is the speed of light, G is the gravitational constant, $\kappa_{\rm IR}$ is the IR opacity, and $\kappa_{\rm UV}$ is the UV opacity.

(c) Within the same host galaxy as in (b), the quasar luminosity L is now such that it exceeds the critical luminosity. Derive the radial velocity profile of the steady-state wind this quasar would drive in both the single-scattering and optically thin UV limit. Discuss if the wind is accelerating or decelerating (at small and large distances) and find the location of the maximum wind velocity for both cases. What is the physical explanation for these results?

[*Hint: You may assume that the wind velocity at the initial launch radius is negligible.*]

END OF PAPER