UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2022 1:30 pm to 4:30 pm

PAPER 346

FORMATION OF GALAXIES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Consider a Universe that consists of just baryons and photons. Show that, during the radiation era, the sound speed of the photon-baryon fluid is

$$c_s = \frac{c}{\sqrt{3}}\left[\frac{3\bar{\rho}_b(z)}{4\bar{\rho}_\gamma(z)} + 1\right]^{-1/2},$$

where $\bar{\rho}_b(z)$ and $\bar{\rho}_\gamma(z)$ are the mean energy densities of baryons and photons at redshift z and c is the speed of light.

The comoving Jeans length of a collisional fluid with homogenous density $\bar{\rho}_b$ and sound speed c_s takes the form

$$\lambda_\mathrm{J}^\mathrm{com} = \frac{c_\mathrm{s}}{a(t)}\sqrt{\frac{\pi}{G\bar{\rho}_b}},$$

where $a(t)$ is the scalefactor. Deduce how the comoving Jeans length $\lambda_\mathrm{J}^\mathrm{com}$ for the baryons behaves from before the epoch of matter-radiation equality t_eq to after the epoch of recombination t_rec.

On a graph of comoving scale versus time, show the evolution of the comoving Jeans length for adiabatic, isentropic perturbations of baryons in an Einstein-de Sitter Universe.

Now suppose the Universe contains cold dark matter particles which can be treated as a collisionless fluid. How does the formula for the comoving Jeans length change?

Derive the behaviour of the comoving Jeans length for cold dark matter particles, distinguishing the behaviour of $\lambda_\mathrm{J}^\mathrm{com}$ in the epochs when the cold dark matter particles (i) are relativistic $t < t_\mathrm{NR}$, (ii) are non-relativistic but still coupled to the photons $t_\mathrm{NR} < t < t_\mathrm{dec}$ and (iii) are fully decoupled $t > t_\mathrm{dec}$. On a new plot, show the behaviour of $\lambda_\mathrm{J}^\mathrm{com}$ from before the epoch of matter-radiation equality t_eq to after the epoch of recombination t_rec.

(b) In a flat Universe with non-zero cosmological constant Λ, the equation governing the evolution of the radius of a shell containing mass M in a spherically symmetric perturbation is

$$\frac{d^2r}{dt^2} = \frac{L^2}{r^3} - \frac{GM}{r^2} + \frac{\Lambda}{3}r.$$

Explain why the angular momentum L is a conserved quantity.

Show that the energy

$$E = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{L^2}{r^2} - \frac{GM}{r} - \frac{\Lambda}{6}r^2,$$

is conserved.

[QUESTION CONTINUES ON THE NEXT PAGE]

Assuming a uniform sphere with mass M and $L = 0$ has turnaround radius $r_{\rm ta}$, show that the potential energies at turnaround due to gravity and Λ are

$$W_{\rm G,ta} = -\frac{3}{5}\frac{GM^2}{r_{\rm ta}}, \qquad W_{\Lambda,\rm ta} = -\frac{1}{10}\Lambda M {r_{\rm ta}}^2.$$

Show that, in the final virialized state, the kinetic energies and potential energies are related by

$$2T_{\rm f} + W_{\rm G,f} = 2W_{\Lambda,\rm f},$$

where $T_{\rm f}$ is the final kinetic energy.

Hence, show that the final radius $r_{\rm f}$ satisfies the cubic equation

$$2\eta(r_{\rm f}/r_{\rm ta})^3 - (2+\eta)(r_{\rm f}/r_{\rm ta}) + 1 = 0,$$

with $\eta = \Lambda/(4\pi G\rho_{\rm ta})$ where $\rho_{\rm ta}$ is the mean density at turnaround.

Show further that

$$\frac{r_{\rm f}}{r_{\rm ta}} \approx \frac{1-\eta/2}{2-\eta/2},$$

Provide a physical interpretation of this result, in particular comparing the repulsive or positive Λ case with the zero Λ case.

2 Suppose a host dark matter halo H_1 contains a subhalo H_2. To a first approximation, the haloes are assumed to be point-like with masses M_1 and $M_2 < M_1$ respectively. They are in circular motion about their common centre of mass O. If the separation of H_1 and H_2 is R, show that the angular velocity of the line H_1H_2 in the centre of mass frame is

$$|\underline{\omega}|^2 = \frac{G(M_1 + M_2)}{R^3}.$$

Let us consider the equilibrium points of a test particle S belonging to subhalo H_2. Let us introduce Cartesian (x, y, z) with origin at O, and with z-axis parallel to $\underline{\omega}$ and x-axis along the line H_1H_2. Give a physical reason why the equilibrium points lie in the plane $z = 0$.

Show that the equilibrium points are the stationary points of

$$E(x, y) = -\frac{\omega^2}{2}|\underline{r}_S|^2 - \frac{GM_1}{|\underline{r}_S - \underline{r}_1|} - \frac{GM_2}{|\underline{r}_S - \underline{r}_2|},$$

where $\underline{r}_S$, $\underline{r}_1$ and $\underline{r}_2$ are the position vectors of S, H_1 and H_2 respectively.

Let $\underline{r}_S = (x, y, 0)$. Using R as the unit of length and $G(M_1 + M_2)/R^2$ as the unit of energy, show that the stationary points on the line joining H_1 and H_2 are the extrema of

$$F(x) = -\frac{x^2}{2} - \frac{1-\alpha}{|x+\alpha|} - \frac{\alpha}{|x+\alpha-1|},$$

where $\alpha = M_2/(M_1 + M_2)$.

Find the extrema in the limit of small α, and show that there are three equilibrium points at:

$$\begin{aligned} L_1 &= \Big(1 - (\alpha/3)^{1/3}, 0, 0\Big), \\ L_2 &= \Big(1 + (\alpha/3)^{1/3}, 0, 0\Big), \\ L_3 &= \Big(-1 - 5\alpha/12, 0, 0\Big). \end{aligned}$$

Are there any other equilibria?

Show that the tidal radius of H_2 is

$$r_t = R\left(\frac{M_2}{3M_1}\right)^{1/3},$$

and provide a physical explanation of the formula.

Explain *qualitatively* the changes to the tidal radius formula if (i) the haloes H_1 and H_2 are extended masses and if (ii) the orbit of H_2 is no longer circular.

The subhalo H_2 contains both stars and dark matter. The length scale of the dark matter is much greater than the stars. Explain why four tidal tails are formed.

3 The density of a cold dark matter halo (the *NFW model*) is

$$\rho(r) = \frac{\rho_c^0 \delta_{\rm char}}{(r/r_s)(1 + r/r_s)^2},$$

where $\delta_{\rm char}$ is a characteristic overdensity. The scale-radius r_s is related to the halo concentration c by $r_s = r_v/c$. Here, r_v is the virial radius which is defined as the distance from the centre of the halo within which the mean density is v times the present critical density ρ_c^0. What is the virial mass M_v of the halo?

Give an order of magnitude estimate of v assuming an Einstein-de-Sitter Universe.

Given that $\delta_{\rm char} = vc^3 g(c)/3$ for some unknown function $g(c)$, show that the enclosed mass is

$$\frac{M(s)}{M_v} = g(c)\left[\ln(1 + cs) - \frac{cs}{1 + cs}\right],$$

where $s = r/r_v$.

Derive the behaviour of the enclosed mass at large and small radii.

If the dark matter particles are warm rather than cold, what qualitative change occurs to the central parts of the halo?

If V_v is the circular velocity at the virial radius, find the form of $g(c)$ by verifying that the gravitational potential of the NFW dark halo is

$$\phi(s) = -g(c) V_v^2 \frac{\ln(1 + cs)}{s}.$$

Show that the circular speed is

$$V^2(s) = V_v^2 \frac{g(c)}{s}\left[\ln(1 + cs) - \frac{cs}{1 + cs}\right].$$

Plot the circular velocity curve of the NFW model for two different concentrations and comment on its structure given the observed flatness of galaxy rotation curves.

Explain physically why NFW halos satisfy a mass-concentration relation.

At the present epoch, let the mass-concentration relation of NFW haloes be

$$c \propto M_v^{-0.1}.$$

Suppose that a $10^{12} M_\odot$ dark matter halo has a virial radius of $r_v = 200$ kpc and a concentration $c = 10$. Estimate the virial radius, scalelength and concentration of a dark halo that is 1024 times less massive.

What astrophysical object might reside in such a dark halo?

4 From the momentum equation for a fluid in proper coordinates, derive the corresponding equation in comoving coordinates as

$$\frac{\partial \underline{v}}{\partial t} + \frac{\dot{a}}{a}\underline{v} + \frac{1}{a}\underline{v} \cdot \nabla \underline{v} = -\frac{1}{a}\nabla\Phi - \frac{1}{a\bar{\rho}(1+\delta)}\nabla P,$$

where a is the scalefactor, $\underline{v}$ is the peculiar velocity, Φ is the peculiar potential, P is the pressure, $\bar{\rho}$ is the mean background density and δ is the overdensity.

Give a physical interpretation of the relationship between the peculiar potential Φ and the proper gravitational potential ϕ.

By linearizing the Euler equations for a pressureless fluid, show that

$$\frac{d}{dt}(a\underline{v}) = -\nabla\Phi.$$

Given that at early times, the Universe behaves like an Einstein-de Sitter cosmology, show that

$$\underline{v} = -\frac{\nabla\Phi_{\mathrm{i}}}{a}\int \frac{D(a)}{a}dt,$$

where $D(a)$ is the linear growth rate. Here, and henceforth, the subscripted roman i refers to an initial or fiducial epoch.

Show that $D(a)$ satisfies

$$\frac{D(a)}{a} = \frac{1}{4\pi G\bar{\rho}_i}\frac{d}{dt}\left(a^2\frac{dD}{dt}\right).$$

Hence, show that the position $x(t)$ of any particle is related to its position at the initial epoch x_{i} by (the *Zeldovich approximation*)

$$x(t) \approx x_{\mathrm{i}} - \frac{D(a)}{4\pi G\bar{\rho}_i}\nabla\Phi_i.$$

Give a physical interpretation of the Zeldovich approximation.

Derive the linearised equation for the evolution of an overdensity in the Zeldovich approximation.

Explain, with an example, how the Zeldovich approximation predicts the formation of caustics in the final density field.

END OF PAPER