UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2022 1:30 pm to 4:30 pm

PAPER 345

FLUID DYNAMICS OF THE ENVIRONMENT

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Starting from equations for the conservation of momentum, mass and volume, show that two-dimensional Boussinesq linear internal gravity waves in an incompressible fluid obey

$$\left[\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\nabla^2 + N^2\frac{\partial^2}{\partial x^2}\right]\psi = 0, \qquad (\star)$$

where ν is the fluid viscosity, κ is the mass diffusivity, N is the constant buoyancy frequency and $\psi = \psi(x, z, t)$ is the streamfunction with x horizontal, z vertically upwards and t time.

(b) Consider a plane wave with its group velocity directed into the first quadrant and introduce ξ and ζ as rotated coordinates such that the unit vector $\hat{\boldsymbol{\xi}}$ is in the direction of the wavenumber vector and $\hat{\boldsymbol{\zeta}}$ is in the direction of the group velocity. Rewrite $(\star)$ in these coordinates.

(c) For the case $\nu = \kappa = \epsilon$ with a plane wave $\psi = \tilde{\psi}(\zeta)e^{i(k\xi-\omega t)}$, determine the leading-order along-wave structure $\tilde{\psi}(\zeta)$ of the streamfunction at high Reynolds number Re, providing a suitable definition of Re. The wavenumber k and frequency ω are constant. [*Hint: Introduce a scaled along-beam coordinate and consider terms at* $\mathcal{O}(\epsilon^0)$ *and* $\mathcal{O}(\epsilon^1)$*).*]

(d) Suppose the free surface and bottom of the ocean are located at $z = 0$ and $z = -H_0(1 - x/L)$, respectively. For what range of ω will the energy of waves propagating to the right remain trapped in $0 < x < L$? Determine γ, the *focusing power* describing how the wavenumber changes due to a reflection from the slope and express this in terms of ω, N, H_0 and L. How does the energy density per wavelength change due to a reflection from the slope?

(e) A ray with a rightward component to its propagation is incident on the origin with an energy density per wavelength of $\bar{E}_0$. Give an expression for ℓ, the distance the wave propagates before reflecting from the sloping bottom. What is x_2, the location where the ray next reflects from the free surface? What is $\bar{E}_2$, the energy density per wavelength at this point? You may ignore any enhanced dissipation or diffusion that may occur due to the presence of the boundaries.

2

A chemical spill creates a high-Reynolds-number flow in a channel of unit width and rectangular cross-section beneath a stationary deep ambient fluid of constant density ρ_0. A reaction between the chemical and the ambient fluid means the density of the flow is given by $\rho = \rho_0 + R_1\phi + R_2\phi^2$, where $0 \leqslant \phi \leqslant 1$ is the concentration of the chemical and R_1, R_2 are constants. Entrainment of ambient fluid *into* the shallow layer is described by the entrainment velocity w_e and detrainment from the shallow layer to the ambient by the detrainment velocity w_d. The detrainment can be assumed not to affect the density of ambient fluid or to induce a motion in it.

(a) State the conditions required for this flow to be treated as shallow water. Assuming the flow is shallow water with depth h and velocity u, derive equations for conservation of the chemical and of conservation of mass for the shallow layer. Show that conservation of momentum can be written in the form

$$\frac{\partial \rho u h}{\partial t} + \frac{\partial}{\partial x}\left(\rho u^2 h + \frac{1}{2}(\rho - \rho_0) g h^2\right) = M,$$

and determine M.

(b) For a Boussinesq flow, express the shallow-water equations in terms of h, u, ϕ and determine the characteristics for the system. Determine also the equation along the intermediate characteristic.

(c) For the special case where $R_1, R_2 \geqslant 0$, w_d is constant and the entrainment velocity adjusts so that the volume of the current is constant, specify an appropriate front condition and derive an integral model for the release of a volume $V(t=0) = V_0$ of the chemical at concentration $\phi(t=0) = 1$. Suppose the length of the current at the time of release is L_0. Determine the relationship between the length of the current $L(t)$ and the concentration $\phi(t)$. Determine the run-out length L_∞ as $t \to \infty$. [*Hint:* $\int \frac{1+af}{\sqrt{f(1+af)}}\, df = \sqrt{f(1+af)} + \frac{1}{\sqrt{a}} \sinh^{-1} \sqrt{af}$.]

3

Consider a room of height H and width X. A radiator is located on the floor against one wall and extends along the length Y of the room. The radiator can be considered as a line source supplying buoyancy flux F_0 (per unit length) with negligible volume or momentum flux. The plume that develops above the radiator can be assumed to have a triangular profile for the velocity and buoyancy such that both are maximal at the wall and decay linearly over a distance $b(z)$ away from the wall. Far from the wall the density in the room is $\hat{\rho}(z)$ with z oriented vertically upwards.

(a) Determine expressions for the volume flux $V(z)$, mass flux $Q(z)$, momentum flux $M(z)$ and buoyancy flux $F(z)$ (all per unit length) in terms of the plume velocity $W(z)$ and density $\rho(z)$ at the wall.

(b) Describe 'Batchelor entrainment' as it may apply to the plume that develops above the radiator. State the Boussinesq approximation and discuss any limitations to applying it to this flow. For a plume with constant F_0, use scaling arguments for the rise-time and volume flux of the plume to show that the plume can be considered quasi-steady when $X \gg H$ despite the stratification in the room changing.

(c) Assuming the flow is steady and Boussinesq, determine also W, b and $g' = \frac{\hat{\rho}-\rho}{\rho_0} g$ in terms of Q, M and F. Here, ρ_0 is a reference density, which you should describe. Derive the equations governing the volume flux, mass flux and momentum flux. Show that conservation of buoyancy flux can be written as

$$\frac{dF}{dz} = -QN^2,$$

where $N = N(z)$ is the vertical profile of the buoyancy frequency.

(d) Suppose that before the radiator is turned on at $t = 0$ the stratification in the room is described by $N(z) = N_0$ for constant N_0. Using sketches and dimensional arguments, describe the initial quasi-steady form of the plume for a range of different N_0.

(e) During the working week, in addition to the radiator supplying heat to the room, a mechanical ventilation system supplies fresh air near floor level at constant temperature T_0 (corresponding to density ρ_0) and volume flux A per unit length of the room. The same volume flux is extracted at high level. The inlet and outlet are both located far from the wall against which the radiator is positioned and have sufficient area that the momentum flux through them is insignificant. Sketch the final steady state that will develop. Assuming the flow is Boussinesq, determine the height of the interface for this steady state.

END OF PAPER