UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 10 June, 2022 1:30 pm to 4:30 pm

PAPER 341

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **THREE** questions from Section A and **ONE** question from Section B.
Each question from Section B carries twice the weight of a question from Section A.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

The ODE $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, $\mathbf{y}(0) = \mathbf{y}_0$, is solved by the s-step method

$$\sum_{k=0}^{s} \rho_k \mathbf{y}_{n+k} = h[\sigma_s \mathbf{f}(\mathbf{y}_{n+s}) + \sigma_{s-1}\mathbf{f}(\mathbf{y}_{n+s-1})],$$

where $\rho_s = 1$, while $\rho_0, \ldots, \rho_{s-1}$ and σ_{s-1}, σ_s are given constants.

a. Derive the conditions that the coefficients σ_k and ρ_k need satisfy so that the method is of order p.

b. Assuming without proof that the highest attainable order of such methods is $s+1$, find such methods for $s = 1$ and $s = 2$. Are they convergent?

c. Characterise all highest-order methods of this kind which are A-stable.

2

Consider the Runge–Kutta method with the Butcher tableau

$$\begin{array}{c|cc} \alpha & \frac{\alpha(2-\alpha)}{2(1-\alpha)} & -\frac{\alpha^2}{2(1-\alpha)} \\ 1 & \frac{1}{2(1-\alpha)} & \frac{1-2\alpha}{2(1-\alpha)} \\ \hline & \frac{1}{2(1-\alpha)} & \frac{-1+2\alpha}{2(1-\alpha)} \end{array},$$

where $\alpha \neq 1$ is a real parameter.

a. Prove that this is a collocation method.

b. Determine the order of the method for different values of α.

c. For which values of α is the method algebraically stable?

3

Consider the linear Schrödinger equation

$$\mathrm{i}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - V(x)u, \qquad -1 \leqslant x \leqslant 1,$$

given with initial conditions at $t = 0$ and periodic boundary conditions. Here $u(x,t)$ is complex-valued while the potential V is a sufficiently smooth real function.

a. Prove that the $L_2[-1,1]$ norm of u is independent of t.

b. The equation is semi-discretised by

$$\mathrm{i}u_m' = \frac{u_{m-1} - 2u_m + u_{m+1}}{(\Delta x)^2} - V(x_m)u_m, \qquad m = -N+1, \ldots, N,$$

where $x_m = m\Delta x$, $\Delta x = 1/N$ and $u_m(t) \approx u(m\Delta x, t)$. Prove that the ℓ_2 norm of the vector $\mathbf{u}$ is independent of t. Is the scheme stable?

c. Propose a second-order time-stepping scheme for the semi-discretisation which also respects the ℓ_2 norm.

4

a. Define a positive-definite operator $\mathcal{L}$.

b. Derive, given a positive-definite, real linear operator $\mathcal{L}$, the variational problem corresponding to the equation $\mathcal{L}u = f$ with f in an appropriate Sobolev space. Prove that this variational principle has a single global minimum and deduce the existence and uniqueness of a weak solution.

c. Prove that the biharmonic operator $\mathcal{L} = \Delta^2$, given in the square $|x|, |y| \leqslant 1$ with zero boundary conditions, is positive definite.

5

a. Let $F(t) = \mathrm{e}^{\Omega(t)}$ be a function from $\mathbb{R}$ to square matrices and suppose that it obeys the identity $F(t)F(-t) \equiv I$ for all $t \in \mathbb{R}$. Prove that Ω is an odd function of t. Such a function F is said to be symmetric.

b. Let $F(t) = \mathrm{e}^{\frac{1}{2}tA}\mathrm{e}^{tB}\mathrm{e}^{\frac{1}{2}tA}$ be the Strang splitting. Prove that it is symmetric and that

$$F(t) = \mathrm{e}^{t(A+B)} + Ct^3 + O(t^4)$$

for some matrix C (which depends on A and B).

c. Let α be a real constant and set $G_\alpha(t) = F(\alpha t)F((1-2\alpha)t)F(\alpha t)$, where F is the Strang splitting. Find a real α so that

$$G_\alpha(t) = \mathrm{e}^{t(A+B)} + O(t^4).$$

d. Proving that also G_α is symmetric, deduce that for the above choice of α actually it is true that

$$G_\alpha(t) = \mathrm{e}^{t(A+B)} + O(t^5).$$

SECTION B

6

Write an essay on stability analysis of Runge–Kutta methods. You should state and prove relevant theorems, explain their importance and accompany your presentation with examples.

7

Write an essay on Fourier stability analysis of time-dependent linear differential equations.

You should introduce all relevant concepts, find a necessary and sufficient condition for stability of finite-difference methods (both fully and semi-discretised) for the Cauchy problem, discuss the connection with Toeplitz operators, specify (and explain) the interplay of stability and boundary conditions and comment on the role of group velocity in stability analysis. You should also briefly (e.g. by an example) describe Fourier analysis of two-step schemes.

END OF PAPER