UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Friday, 10 June, 2022 9:00 am to 11:00 am

## PAPER 336

## PERTURBATION METHODS

### Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

**STATIONERY REQUIREMENTS**
Cover sheet
Treasury tag
Script paper
Rough paper

**SPECIAL REQUIREMENTS**
None

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** (a) Find the first two nonzero terms in the asymptotic expansion of the large positive real root of the equation

$$\exp(-x^2) = \epsilon x$$

in the limit $\epsilon \to 0$.

(b) Consider

$$I(\epsilon) = \int_0^1 \frac{\exp(x)\mathrm{d}x}{\sqrt{\epsilon + x}} \,.$$

Show that

$$I(\epsilon) \sim I(0) - 2\sqrt{\epsilon} + [\mathrm{e} - I(0)]\epsilon + O(\epsilon^{3/2})$$

as $\epsilon \to 0$.

(c) Find the asymptotic behaviour of

$$J_\nu(\nu \operatorname{sech} \alpha) = \frac{1}{2\pi \mathrm{i}} \int_{\infty - \mathrm{i}\pi}^{\infty + \mathrm{i}\pi} \exp(\nu \operatorname{sech} \alpha \sinh t - \nu t)\mathrm{d}t$$

for real $\nu$ and $\alpha$, as $\nu \to \infty$ with (i) $\alpha > 0$, (ii) $\alpha = 0$. In (ii) you should write your answer in terms of the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t)\mathrm{d}t \,.$$

In each case the full details of the steepest descent contour need not be considered.

**2** (a) The oscillation of a simple pendulum with weak cubic damping is described by the governing equation

$$\ddot{x} + \epsilon|\dot{x}|^2\dot{x} + x = 0 \ ,$$

with $x(0) = 1$ and $\dot{x}(0) = 0$. In the limit $\epsilon \to 0$, use the method of multiple scales to find the leading-order solution valid up to and including times $t = O(1/\epsilon)$.

What happens up to and including times $t = O(1/\epsilon)$ when the governing equation is replaced by

$$\ddot{x} + \epsilon \sin(\dot{x})\dot{x} + x = 0 \ ,$$

with the same initial conditions? [Hint: you may find it useful to use the power series expansion of $\sin(\dot{x})$.]

(b) Consider the propagation of sound waves along a two-dimensional duct, the width of which varies slowly in the axial direction: specifically the duct is aligned in the $x$ direction and the walls of the duct are given by $y = \pm h(\epsilon x)$, where $\epsilon \ll 1$. The unsteady flow is described by the velocity potential

$$\phi(x, y) \exp(\mathrm{i}\omega t) \ ,$$

which satisfies the Helmholtz equation

$$\nabla^2\phi + k_0^2\phi = 0$$

subject to Dirichlet boundary conditions on the walls , i.e. $\phi(x, \pm h) = 0$.

In the case $h = h_0$ constant, show that the general solution of the problem takes the form

$$\sum_{n=1}^{\infty} [A_n \exp(\mathrm{i}k_n x) + B_n \exp(-\mathrm{i}k_n x)] \sin(n\pi y/h_0) \ ,$$

where $A_n, B_n$ are arbitrary constants and the wavenumbers $k_n$ are to be determined.

Now consider the case of $h = h(X)$ varying, where $X = \epsilon x$. Use the method of multiple scales to show that the general solution now takes the form

$$\sum_{n=0}^{\infty} [A_n(X) \exp(\mathrm{i}\Theta_n) + B_n(X) \exp(-\mathrm{i}\Theta_n)] \sin(n\pi y/h) \ ,$$

where

$$\Theta_n(X) = \frac{1}{\epsilon}\int_0^X k_n(\xi)\mathrm{d}\xi$$

and $A_n(X)$ and $B_n(X)$ satisfy first-order ordinary differential equations which you should determine. Solve these equations to determine $A_n(X)$ and $B_n(X)$ in terms of reference values $A_n(0)$ and $B_n(0)$. [NB: you need not consider the case when $k_n(X)$ vanishes.]

**3**

(a) Consider the equation

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (1+x)\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$$

with $y(0) = y(1) = 1$.

Find the first two nonzero terms in the inner and outer expansions in the limit $\epsilon \to 0$. Calculate a uniformly-valid approximation to $y$ which is correct up to and including $O(\epsilon)$.

If the boundary conditions are now replaced by $y(-1) = y(1) = 1$ explain briefly, without detailed calculation, how the structure of the asymptotic solution would change.

(b) Consider

$$(x + \epsilon f)f' + f = 2x \quad \text{when } 0 \leqslant x \leqslant 1$$

with $f(1) = 2$, in the limit $\epsilon \to 0$.

You are given that when $x = O(1)$

$$f(x) \sim \frac{1+x}{x} - \epsilon\left(\frac{(1-x^2)^2}{2x^3}\right).$$

Identify the size of the inner region, and calculate the first two nonzero terms in this inner region.

## END OF PAPER