UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2022 9:00 am to 12:00 pm

## PAPER 333

## FLUID DYNAMICS OF CLIMATE

**Before you begin please read these instructions carefully**

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

Cartesian co-ordinates $(x, y, z)$ are used with $z$ denoting the upward vertical. The corresponding velocity components are $(u, v, w)$. Unless stated otherwise, $g$ is the gravitational acceleration.

**STATIONERY REQUIREMENTS**
Cover sheet
Treasury tag
Script paper
Rough paper

**SPECIAL REQUIREMENTS**
None

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** The linearised equations for shallow-water motion on an $f$-plane are:

$$u_t - fv = -\phi_x \tag{1}$$
$$v_t + fu = -\phi_y \tag{2}$$
$$\phi_t + c^2(u_x + v_y) = 0, \tag{3}$$

where $f$ is a constant.

(i) Give a brief physical interpretation of each equation.

(ii) Show that the potential vorticity $P = v_x - u_y - f\phi/c^2$ satisfies $P_t = 0$.

(iii) Show that plane-wave disturbances with wavenumber $(k, l)$ can have three possible frequencies corresponding to the roots of

$$\omega(\omega^2 - \Omega^2(k, l)) = 0$$

where the form of $\Omega(k, l)$ should be derived. Interpret the form of the dispersion relation, including the form of $\Omega(k, l)$ when $(k^2 + l^2)^{1/2}$ is small or large, with the meaning of 'small' and 'large' to be explained, and the corresponding separation of $u$, $v$ and $\phi$ into zero-frequency ($\omega = 0$) and oscillatory ($\omega = \pm\Omega(k, l)$) parts.

(iv) Use the results determined above to give arguments why, for an initial condition $u(x, y, 0) = 0$, $v(x, y, 0) = 0$, $\phi(x, y, 0) = \phi_0 \mathrm{sgn}(x)$ there is an adjustment to a steady state $\phi = \Phi(x, y)$, with $\Phi(x, y)$ satisfying the equation

$$\Phi_{xx} + \Phi_{yy} - (f^2/c^2)\Phi = (-f^2/c^2)\phi_0 \mathrm{sgn}(x). \tag{4}$$

Why at a large but finite value of $t$ does this solution only apply for a finite range of $x$?

Now and for the remainder of the question consider the case where the fluid is confined between rigid boundaries at $y = 0$ and $y = L$.

(v) The equation for the steady adjusted state in this case is (4) but additional boundary conditions are needed. One is $\phi_x = 0$ on $y = 0$ and $y = L$. Why? This condition by itself is not sufficient to determine a unique solution. Show, giving the required conditions on $\Psi$, that a function $\Psi(y)$ can be added to any solution and still satisfy the equation and the $\phi_x = 0$ boundary condition.

(vi) Show that with the rigid boundaries the equations have additional 'Kelvin-wave' solutions in $0 < y < L$ with $v = 0$, $u = F_\pm(x \pm ct)\tilde{u}_\pm(y)$, $\phi = F_\pm(x \pm ct)\tilde{\phi}_\pm(y)$, where the forms of $\tilde{u}_\pm(y)$ and $\tilde{\phi}_\pm(y)$ are to be determined.

(vii) Show also that the presence of the boundaries implies that

$$\left(\frac{\partial}{\partial t} \pm \frac{1}{c}\frac{\partial}{\partial x}\right) \int_0^L e^{\mp fy/c}\left(u + \frac{1}{c}\phi\right) dy = 0. \tag{5}$$

(viii) Use (5) to deduce additional conditions on the solution of (4) that determine a unique solution for the steady adjusted state.

**2** (a) Consider steady horizontal flow, uniform in $x$ and $y$, on an $f$-plane and above a flat surface $z = 0$. The fluid has a constant kinematic viscosity $\nu$ implying that the horizontal momentum equations contain a term $\nu(\partial^2 u/\partial z^2, \partial^2 v/\partial z^2)$, representing the viscous force per unit mass, and a non-slip condition is applied at $z = 0$. Far above the surface the horizontal flow is equal to $(U, 0)$ and is in geostrophic balance.

Solve the steady momentum equations to obtain the horizontal velocity $(u(z), v(z))$. What is the vertical length scale $\delta$ on which the horizontal velocity varies? Calculate the vertically integrated horizontal transport $\mathbf{u}_{\mathrm{T}}$ by the flow anomaly $(u(z) - U, v(z))$. How does this relate to the vertically integrated momentum balance for the fluid?

Assuming that your results apply to the case where the horizontal velocity far above the surface varies slowly in $x$ and $y$ deduce an expression for $\nabla.\mathbf{u}_{\mathrm{T}}$. What is the resulting effective boundary condition on the vertical velocity $w$ seen by the interior flow for $z \gg \delta$?

(b) Now consider quasi-geostrophic flow on the $f$-plane with constant buoyancy frequency $N$. (You do not have to derive the quasi-geostrophic equations but should state clearly any properties of quasi-geostrophic flow that you use.) Explain, with reference to the form of the buoyancy/density equation, why the leading-order approximation to the vertical velocity $w$ is given by

$$w = -\frac{f}{N^2}\frac{D_g \psi_z}{Dt}$$

where $\psi$ is the quasi-geostrophic stream function and $D_g/Dt$ denotes the rate of change following the geostrophic flow.

Using the effective boundary condition derived in (a), give the equations governing the time evolution of quasi-geostrophic flow of a viscous fluid above a rigid boundary at $z = 0$, assuming that $\nu$ is small enough that viscosity can be neglected in the interior flow and that $\delta$ is small enough that the effective boundary condition can be applied at $z = 0$. (Your equations should include one that applies in the interior, $z > 0$ and one that applies on the boundary $z = 0$.)

Assume that the equations can be linearised about a state of rest and solve for the evolution of $\psi(x, y, z, t)$ when $\psi(x, y, z, 0) = \psi_0 \sin(kx)$, i.e. the flow is initially independent of $y$ and $z$.

Describe the evolution, including the dependence on $f$, $N$ and the horizontal scale $k^{-1}$ of the initial flow anomaly. Deduce that the effect of the lower boundary is felt only a finite distance into the fluid. Comment on the reasons for this.

**3** When linearised about a uniform flow $(U,0,0)$ the quasi-geostrophic potential vorticity equation on a $\beta$-plane has the form

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)q'+\beta\frac{\partial}{\partial x}\psi'=Q' \tag{1}$$

where $q'=\psi'_{xx}+\psi'_{yy}+(f_0^2/N^2)\psi'_{zz}$ with $f_0$ the Coriolis parameter and $N$ the buoyancy frequency (assumed constant). $Q'$ represents the effect of forcing or dissipation processes.

From the equation above derive the Eliassen-Palm wave activity relation

$$\frac{\partial\overline{\mathcal{A}}}{\partial t}+\nabla.\overline{\mathbf{F}}=\frac{\overline{Q'q'}}{\beta}, \tag{2}$$

where $\overline{\mathcal{A}}=\overline{q'^2}/2\beta$ and $\overline{\mathbf{F}}=(0,-\overline{u'v'},-gf_0\overline{v'\rho'}/\rho_0N^2)=(0,\overline{F}^{(y)},\overline{F}^{(z)})$.

In the expressions above $\overline{(\,)}$ indicates an $x$-average and $(\,)'$ denotes the difference from the average, i.e. $\chi'=\chi-\overline{\chi}$ for any quantity $\chi$. In (1) the various quantities are assumed to have zero $x$-average by suitable choice of $Q'$. You may assume $u'=-\psi'_y$, $v'=\psi'_x$ and $\rho'=-f_0\rho_0\psi'_z/g$. [Hint: You may find it helpful to recall results, for an arbitrary function $\eta$, such as $\overline{\eta_x}=0$ and corollaries such as $\overline{\eta_x\eta_{yy}}=(\overline{\eta_x\eta_y})_y-\overline{\eta_{xy}\eta_y}=(\overline{\eta_x\eta_y})_y$. ]

Explain the relation of each of the terms in (2) to the forcing, propagation and dissipation of waves.

The term $\nabla.\overline{\mathbf{F}}$ can also be shown to be equivalent to the $x$-average force in the $x$-direction due to the waves. Explain why this force is zero if the waves are steady and there is no forcing or dissipation (i.e. the term $Q'$ in (1) is zero).

Now consider the explicit case of a background flow $(U,0,0)$ in $z>0$ confined between rigid boundaries at $y=0$ and $y=L$, with $U>0$. Waves are forced by a steady topographic perturbation $h'(x,y)=\Re(\hat{h}e^{ikx})\sin ly$ at the lower boundary, with $l=\pi/L$ and $k>0$. There is thermal damping acting, implying that $Q'$ in (1) is given by $Q'=-\alpha f_0^2\psi'_{zz}/N^2$ where $\alpha>0$. Show that there is a solution for $\psi'$ of the form $\Re\left(\hat{\psi}(z)e^{ikx}\right)\sin ly$, where $\hat{\psi}(z)$ satisfies the ordinary differential equation

$$\frac{f_0^2}{N^2}\frac{d^2\hat{\psi}}{dz^2}-\left(k^2+l^2\right)\hat{\psi}+\frac{\beta}{U}\hat{\psi}=\frac{i\alpha}{kU}\frac{f_0^2}{N^2}\frac{d^2\hat{\psi}}{dz^2}. \tag{3}$$

The lower boundary condition implies that $d\hat{\psi}/dz=C(k,U,\alpha)\hat{h}$ at $z=0$, where $C(k,U_0,\alpha)$ is complex.

Show that (3) has a solution such that $|\hat{\psi}(z)|\to 0$ as $z\to\infty$, whatever the sign of $(\beta/U)-k^2-l^2$. Distinguish between the forms of the solution for $0<U<\beta/(k^2+l^2)$ and for $0<\beta/(k^2+l^2)<U$. [ Hint: You may find it helpful to write $(1-i\alpha/kU)^{-1/2}=\gamma_r+i\gamma_i$ where (since $k$, $\alpha$ and $U$ are all assumed positive in this case) $\gamma_r>0$ and $\gamma_i>0$. ]

Show that for this solution $\overline{F}^{(y)}=0$. Evaluate $\overline{F}^{(z)}$ and deduce the form of the $x$-average force exerted by the waves. Comment on the implication for the force exerted by the flow on the lower boundary topography.

**4** Consider the equations for shallow-water motion on an equatorial $\beta$-plane, assuming that geostrophic balance holds in the $y$-momentum equation and with frictional and thermal damping terms included respectively in the $x$-momentum and thickness equations:

$$u_t - \beta y v = -\phi_x - \gamma u \tag{1}$$

$$\beta y u = -\phi_y \tag{2}$$

$$\phi_t + c^2(u_x + v_y) = -\alpha\phi. \tag{3}$$

The constants $\alpha$ and $\gamma$ are both positive.

Consider plane-wave equatorially trapped solutions of the form $u = \Re(\hat{u}(y)e^{i(kx-\omega t)})$, $v = \Re(\hat{v}(y)e^{i(kx-\omega t)})$, $\phi = \Re(\hat{\phi}(y)e^{i(kx-\omega t)})$. [You may assume that the eigenvalue problem $V_{yy} - y^2 V = \lambda V$ with $|V| \to 0$ as $|y| \to \infty$ has eigenvalues $\lambda_n = -(2n+1)$ for $n = 0, 1, \ldots$ with corresponding eigenfunctions $V_n(y) = H_n(y)\exp(-y^2/2)$, where the $H_n(.)$ are the Hermite polynomials, with $H_0(y) = 1$, $H_1(y) = 2y$, $H_2(y) = 4y^2 - 2$, etc.]

(i) Show that there is a Kelvin-wave solution with $\hat{v}(y) = 0$. Give the dispersion relation $\omega(k)$ and the form of each of the functions $\hat{u}(y)$ and $\hat{\phi}(y)$. Note any restrictions on $k$ for equatorially trapped solutions to exist.

(ii) Show that there is a family, labelled by $n = 1, 2, \ldots$ of Rossby-wave solutions with $\hat{v}(y) \neq 0$. Give the dispersion relation (as a function of $n$) and the corresponding form of the function $\hat{v}(y)$. Note any restrictions on $k$ for equatorially trapped solutions to exist. [Hint: Defining a new variable $Y = \sigma y$ where $\sigma$ is a suitably chosen constant, in general complex, may be helpful.]

In each of your answers above, provide careful explanation where some roots of an algebraic equation are rejected and some are retained.

(iii) Now consider the case where $\omega$ is given, and specified to be zero, and the (complex) value of $k$ is to be deduced. This gives insight into the $x$-structure of the response to a localised forcing. You may assume that $\Im(k) > 0$ implies a response that appears to the east of the forcing and decays with distance away from the forcing region and that $\Im(k) < 0$ correspondingly implies a response that appears to the west of the forcing and decays with distance away from it.

Find the possible values of $k$ from the dispersion relations obtained in (i) and (ii) above. Use these to describe the structure of the response to a forcing localised on the equator near $x = 0$. Comment in particular on the decay scales of the response to the east and to the west of the forcing region and on the latitudinal scale of the response, including the dependence on $\alpha$ and on $\gamma$.

## END OF PAPER