UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2022 1:30 pm to 4:30 pm

PAPER 329

SLOW VISCOUS FLOW

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Use the Papkovich–Neuber representation for Stokes flow, explaining your choice of potential, to determine the flow due to a couple $\mathbf{G}$ acting on a rigid sphere, radius a centred at $\mathbf{x} = \mathbf{0}$, in unbounded fluid.

(b) Calculate the vorticity of the flow $\mathbf{u}(\mathbf{x}) = -\mathbf{x}\cdot\mathbf{A}\cdot\mathbf{x}\,\mathbf{x}/r^5$ due to a stresslet of strength $8\pi\mu\mathbf{A}/3$.

(c) Two force-free rigid spheres of radius a, denoted by $i = 1, 2$, in unbounded fluid are subject to applied couples $\mathbf{G}_i$ and respond with rigid-body motions $\mathbf{U}_i + \boldsymbol{\Omega}_i \wedge \mathbf{x}$ relative to their centres. Let $\mathbf{R}$ denote the vector distance from sphere 1 to sphere 2.

(i) Briefly explain why if $\mathbf{G}_2 = \mathbf{0}$ the velocities of the two spheres must have the form

$$\mathbf{U}_i = \alpha_i\, \mathbf{G}_1 \wedge \mathbf{R}/(\mu a^3)\,, \quad i = 1, 2$$

for some functions $\alpha_1(R/a)$ and $\alpha_2(R/a)$ (which need not be determined).

What is the analogous result for the angular velocities $\boldsymbol{\Omega}_i$?

You are now given that $R \gg a$. For convenience, let $\boldsymbol{\Omega}$ denote $\mathbf{G}_1/8\pi\mu a^3$.

(ii) Let $\mathbf{G}_2 = \mathbf{0}$. Find the leading-order approximations to $\mathbf{U}_2$ and $\boldsymbol{\Omega}_2$.

Find the leading-order term in the perturbation flow due to sphere 2 and deduce that the leading-order correction to $\boldsymbol{\Omega}_1$ is given by

$$\boldsymbol{\Omega}_1 - \boldsymbol{\Omega} = -\frac{15a^6}{4R^6}\left(\boldsymbol{\Omega} - \frac{\boldsymbol{\Omega}\cdot\mathbf{R}\,\mathbf{R}}{R^2}\right).$$

(iii) Now let $\mathbf{G}_2$ be such that $\boldsymbol{\Omega}_2 = \mathbf{0}$. Show that the leading-order approximation to $\mathbf{U}_1$ is

$$\mathbf{U}_1 = -\frac{a^6\boldsymbol{\Omega} \wedge \mathbf{R}}{2R^6}.$$

Briefly explain why the next correction to $\mathbf{U}_2$, beyond the leading-order answer to part (ii), is $O(\Omega a^9/R^8)$. [Do not attempt to calculate it!]

[*You may assume the Faxén formulae*

$$\mathbf{U} = \frac{\mathbf{F}}{6\pi\mu a} + \mathbf{u}_\infty + \frac{a^2}{6}\nabla^2\mathbf{u}_\infty\,, \quad \boldsymbol{\Omega} = \frac{\mathbf{G}}{8\pi\mu a^3} + \frac{1}{2}\boldsymbol{\omega}_\infty\,,$$

but should explain how you apply them. You may also assume that the full perturbation flow produced by adding a rigid sphere to a uniform straining flow $\mathbf{u} = \mathbf{E}\cdot\mathbf{x}$ *is given by*

$$\mathbf{u}'(\mathbf{x}) = -\frac{5}{2}\mathbf{x}\cdot\mathbf{E}\cdot\mathbf{x}\,\mathbf{x}\left(\frac{a^3}{r^5} - \frac{a^5}{r^7}\right) - \mathbf{E}\cdot\mathbf{x}\frac{a^5}{r^5}\,.$$]

2

The rupture of a thin sheet of viscous fluid due to an attractive (van der Waals) force per unit area between the two surfaces of the sheet can be analysed in a similar manner to the Rayleigh instability:

Let the sheet occupy the region $-h(x,t) \leqslant y \leqslant h(x,t)$. Assume that the effect of the attraction is simply to modify the usual stress boundary condition on each interface to

$$[\boldsymbol{\sigma}\cdot\mathbf{n}]_-^+ = \left(\frac{V}{h^3} + \gamma\kappa\right)\mathbf{n},$$

where $V > 0$ is a constant coefficient of attraction, γ is the constant coefficient of surface tension, $\mathbf{n}$ is the normal to the interface and κ is its curvature. Gravity and the surrounding air should be neglected.

Consider small symmetric perturbations to a uniform thickness h_0 such that $h = h_0 + \eta(x,t)$, where $\eta \propto \exp(\mathrm{i}kx + st)$, $|\eta| \ll h_0$ and $|\eta_x| \ll 1$. Obtain the linearized boundary conditions at $y = h_0$. Using Papkovich–Neuber potentials with the appropriate symmetry in y, deduce that the growth rate of the rupture instability is given by

$$s = \frac{3V}{\mu h_0^3}\frac{(1-\Gamma K^2)\sinh^2 K}{K(2K+\sinh 2K)}\ , \tag{1}$$

where $K = kh_0$, and identify the constant Γ.

Sketch the form of $s(K)$ for the cases $\Gamma = 0$ and $\Gamma = 1$, and comment on the physical interpretation of the long and short wavelength behaviour. What additional physical effects might modify your prediction of the most unstable wavelength?

The surfaces of a planar soap film are uniformly covered with surfactant, which reduces the unperturbed coefficient of surface tension to a value γ_0. Use physical arguments, with diagrams, to explain why the surfactant decreases the growth rate of the rupture instability relative to that of a surfactant-free sheet with constant surface tension γ_0. [*Mathematical analysis is* **not** *required.*]

In the limit of strong surfactant effects, (1) is replaced by

$$s = \frac{3V}{\mu h_0^3}\frac{(1-\Gamma K^2)(\sinh 2K - 2K)}{4K\cosh^2 K}\ .$$

For the case $\Gamma \gg 1$ find the maximum growth rate. Comment on the lifetime of a soap bubble.

3

A rigid cylindrical tube, radius a, contains fluid of viscosity μ and a force-free, couple-free rigid sphere with radius b and centre at distance c (with $b + c < a$) from the axis of the tube. Far ahead of and behind the sphere there is uniform Poiseuille flow. Explain why:

(i) c is constant as the sphere is carried along by the flow;

(ii) $a \displaystyle\int_{x_1}^{x_2} \int_0^{2\pi} \sigma_{rx}(a, \theta, x)\, d\theta\, dx = \pi a^2 [p(x_2) - p(x_1)]$,

where (r, θ, x) are cylindrical polar coordinates aligned with the tube, and x_1 and x_2 are two positions far from the sphere.

Consider the case $b = (1 - \epsilon)a$ and $c = \epsilon \lambda a$, where $\epsilon \ll 1$ and $0 \leqslant \lambda < 1$. Work in the frame moving with the sphere. Let the walls of the tube have velocity $-U$, and assume that the angular velocity of the sphere is negligible. The coordinates are chosen such that the width of the narrow gap between the sphere and the tube can be approximated by

$$h(\theta, x) = h_0(\theta) + x^2/(2a), \quad \text{where} \quad h_0 = \epsilon(1 + \lambda \cos\theta)a.$$

Use scaling arguments to estimate the typical magnitudes of (a) the pressure gradient, the pressure and the shear stress in the narrow gap and (b) the pressure gradient and the shear stress ahead of and behind the sphere.

Show that in the gap

$$\left.\frac{\sigma_{xy}}{\mu}\right|_{y=0} = \frac{4U}{h} + \frac{6q}{h^2}, \quad \text{where} \quad q = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x} - \frac{Uh}{2} \quad \text{and} \quad y = a - r,$$

and find a similar expression for σ_{xy} on $y = h$. Explain carefully why q is approximately independent of x.

By considering (ii) at $O(\epsilon^{-3/2})$, show that $q = -\frac{2}{3}Uh_0(\theta)$. Deduce that the pressure gradient far from the sphere is approximately

$$8\mu U(1 - \tfrac{4}{3}\epsilon)/a^2.$$

[*You may assume that if* $I_n \equiv \displaystyle\int_{-\infty}^{\infty} \frac{d\xi}{(1+\xi^2)^n}$ *then* $I_1 = \pi$, $I_2 = \dfrac{\pi}{2}$ *and* $I_3 = \dfrac{3\pi}{8}$. *You may also assume that the volume flux in Poiseuille flow is* $(\pi a^4/8\mu)\partial p/\partial x$.]

By considering (ii) at $O(\epsilon^{-1/2})$, show further that the leading-order pressure drop across the sphere is

$$\sqrt{\frac{2}{\epsilon}}\frac{2\mu U}{a}\int_0^{2\pi} \frac{d\theta}{\sqrt{1 + \lambda\cos\theta}}.$$

Show that $\displaystyle\int \sigma_{xy}|_{y=h}\, dx = 0$ and comment on the significance of this result.

END OF PAPER