UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 7 June, 2022 1:30 pm to 3:30 pm

PAPER 327

DISTRIBUTION THEORY AND APPLICATIONS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define the space of Schwartz functions $\mathcal{S}(\mathbf{R}^n)$ and the space of tempered distributions $\mathcal{S}'(\mathbf{R}^n)$, specifying the notion of convergence in each. Show that a linear form $u : \mathcal{S}(\mathbf{R}^n) \to \mathbf{C}$ belongs to $\mathcal{S}'(\mathbf{R}^n)$ iff $\langle u, \varphi_m \rangle \to 0$ for each sequence φ_m that tends to zero in $\mathcal{S}(\mathbf{R}^n)$.

(b) Define the Fourier transform on $\mathcal{S}(\mathbf{R}^n)$ and $\mathcal{S}'(\mathbf{R}^n)$.

(i) For $\varphi \in \mathcal{S}(\mathbf{R}^n)$ define the *dilation by* $t > 0$ by $\delta_t \varphi(x) = \varphi(tx)$. Using a duality argument, show that this definition extends to $u \in \mathcal{S}'(\mathbf{R}^n)$ via

$$\langle \delta_t u, \varphi \rangle = t^{-n} \left\langle u, \delta_{1/t} \varphi \right\rangle \quad \forall \varphi \in \mathcal{S}(\mathbf{R}^n)$$

and verify that $\delta_t u \in \mathcal{S}'(\mathbf{R}^n)$.

(ii) Call $u \in \mathcal{S}'(\mathbf{R}^n)$ *homogeneous of degree* σ if $\delta_t u = t^\sigma u$ for $t > 0$. Show that if $u \in \mathcal{S}'(\mathbf{R}^n)$ is homogeneous of degree σ then $\hat{u}$ is also homogeneous and find its degree of homogeneity.

(c) For $0 < \alpha < n$ consider $u_\alpha \in \mathcal{S}'(\mathbf{R}^n)$ defined by

$$\langle u_\alpha, \varphi \rangle = \int |x|^{-\alpha} \varphi(x) \, \mathrm{d}x.$$

Show that $\hat{u}_\alpha(\lambda) = c_\alpha |\lambda|^{-\beta}$ where β is a constant you should determine and

$$c_\alpha = \frac{2^{n-\alpha} \Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \pi^{n/2}$$

where $\Gamma(\omega) = \int_0^\infty t^{\omega - 1} e^{-t} \, \mathrm{d}t$ for $\omega > 0$.

[*You will find it useful to use the identity*

$$|x|^{-\alpha} = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty \tau^{\alpha/2} e^{-\tau |x|^2} \, \frac{\mathrm{d}\tau}{\tau}$$

and suitably interchange orders of integration.]

2 Let $X \subset \mathbf{R}^n$ be open. Define the spaces $\mathcal{E}(X)$ and $\mathcal{E}'(X)$, specifying the notion of convergence on each.

(a) Show that for $u \in \mathcal{E}'(\mathbf{R}^n)$, the Fourier transform $\hat{u} \in \mathcal{S}'(\mathbf{R}^n)$ can be identified with the smooth function $\lambda \mapsto \hat{u}(\lambda) = \left\langle u(x), e^{-\mathrm{i}\lambda\cdot x}\right\rangle$ and show that there exists an $N \geqslant 0$ such that $|\hat{u}(\lambda)| \lesssim \langle\lambda\rangle^N$.

Hence prove that for each $v \in \mathcal{E}'(X)$, there exists a finite collection of continuous functions with compact support, $\{f_\alpha\}$, such that

$$v = \sum_\alpha \partial^\alpha f_\alpha \quad \text{in } \mathcal{E}'(X).$$

(b) Let $\delta_0 \in \mathcal{D}'(\mathbf{R})$ denote the Dirac delta supported at zero. Establish the following:

(i) If $\rho \in \mathcal{D}(\mathbf{R})$ with $\rho(0) = 1$ then $\delta_0 = \rho\delta_0$;

(ii) If H denotes the Heaviside function, then $\delta_0 = (xH)''$;

(iii) If $\varphi \in \mathcal{D}(\mathbf{R})$ and $u \in \mathcal{D}'(\mathbf{R})$ then $\varphi u'' = \varphi'' u - 2[\varphi' u]' + [\varphi u]''$.

Using parts (i)–(iii), find explicit examples of continuous functions with compact support, f_0, f_1, f_2, such that

$$\delta_0 = f_0 + f_1' + f_2'' \quad \text{in } \mathcal{D}'(\mathbf{R}).$$

3

State and prove the Malgrange-Ehrenpries theorem. Your proof should involve the construction of an element of $\mathcal{D}'(\mathbf{R}^n)$ which depends on a suitable "Hörmander staircase".

Provide *explicit* Hörmander staircases for the following partial differential operators in two variables, justifying your answers:

$$\text{(i)}\ \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}; \quad \text{(ii)}\ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}; \quad \text{(iii)}\ \frac{\partial}{\partial x} - \frac{\partial^2}{\partial y^2}.$$

You are not required to compute the corresponding fundamental solutions.

END OF PAPER