UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 13 June, 2022 1:30pm to 3:30 pm

PAPER 326

INVERSE PROBLEMS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **BOTH** questions.
There are **TWO** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Generalised solutions

Let $A : \mathcal{X} \to \mathcal{Y}$ be a linear bounded operator and $\mathcal{X}, \mathcal{Y}$ Hilbert spaces. In this exercise, we denote the range and null space of an operator as $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$. You may use results from the lectures as long as they are stated in full.

(a) State the range of the *Moore-Penrose inverse of* A and the *Moore-Penrose equations.*

(b) Prove or falsify the following statements for $A \in \mathbb{C}^{m\times n}$ with $m > n$ and $B \in \mathbb{C}^{n\times r}$. Here A^H denotes the conjugate transpose of A (the matrix formed by conjugating each element and taking the transpose). If there is any statement that is false, give a sufficient condition on the ranks of A and/or B so that the statement becomes true.

(i) $(A^\dagger)^H = (A^H)^\dagger$,

(ii) $(A^H A)^\dagger = A^\dagger (A^\dagger)^H$,

(iii) $\mathrm{rank}(A^\dagger A) = \mathrm{rank}(A^\dagger) = \mathrm{tr}(A^\dagger A)$,

(iv) $(AB)^\dagger = B^\dagger A^\dagger$.

(c) The space $\ell^2(\mathbb{R})$ consists of real-valued infinite sequences $f{=}(f_1, f_2, ...)$ for which $\sum_{j=1}^\infty f_j^2 < \infty$. It is a Hilbert space under the inner product $\langle f, g\rangle_{\ell^2} = \sum_{j=1}^\infty f_j g_j$ and induced norm $\|f\|_{\ell^2} = \sqrt{\sum_{j=1}^\infty f_j^2}$.

(i) Prove that the diagonal operator $D : \ell^2 \to \ell^2$ defined as $[Df]_j = d_j f_j$ with $d_j \in \mathbb{R}$ for $j = 1, ...$ is bounded if and only if $B = \sup_j |d_j| < \infty$.

(ii) Prove that, if the diagonal operator in (i) is bounded, then $\|D\|_{\ell^2} = B$.

(iii) Compute the Moore-Penrose inverse of D. Is $D^\dagger$ continuous?

(iv) Consider D as defined in (i) where $d_j = 1/j$ for $j = 1, ...$ Show that a solution of $Du = f$, for $f \in \ell^2(\mathbb{R})$, might not exist. Show that if a solution exists, it is unique.

(v) Prove that the solution of $Du = f$ for D defined in (iv) is not stable with respect to perturbations in f. To do this, prove or falsify the following property: let $Du^* = f^*$ and $\{u_n\}_{n=1,..}$ be a sequence such that $Du_n = f_n$, then $u_n \to u^*$ if $f_n \to f^*$.

2 Classical Regularization and Functionals

(a) Let $\mathcal{X}$ be a Banach space with topology $\tau_{\mathcal{X}}$. Assume $f : \mathcal{X} \to \bar{\mathbb{R}}$.

(i) Define *sequential lower semi-continuity.*

(ii) Prove that $f(x)$ is a lower semi continuous function if and only if the set $\Omega = \{x \in \mathcal{X} : f(x) \leqslant \eta\}$ is closed.

(iii) Prove that $\mathcal{X}' \subset \mathcal{X}$ is closed implies that the characteristic function $\chi_{\mathcal{X}'}$ of $\mathcal{X}'$ is lower semi continuous.

(b) Consider $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ with dense range $\mathcal{R}(A) \subset \mathcal{Y}$ for $\mathcal{X}$ and $\mathcal{Y}$ Hilbert spaces.

(i) Assume that $A^*A + \alpha I$ is invertible. Derive a lower and an upper bound for $\langle f, (A^*A + \alpha I)f\rangle_{\mathcal{X}}$ and use them to prove that for all $\alpha > 0$ $\|(A^*A + \alpha I)^{-1}\|| \leqslant 1/\alpha$.

(ii) Prove that if $\{z_n\}_n \subset \mathcal{R}(A^*A + \alpha I)$ is a Cauchy sequence then $\{u_n\}_n$ such that $(A^*A + \alpha I)u_n = z_n$ also form a Cauchy sequence with limit $u \in \mathcal{X}$. A sequence $\{z_n\}_n$ is a Cauchy sequence in a metric $d(\cdot, \cdot)$ if for all real $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(z_m, z_n) < \epsilon$ for all $m, n > N$.

(iii) Prove that $A^*A + \alpha I$ is invertible. [*Hint: You can use (ii) as a statement and the fact that injective linear operators have a dense range.*]

(iv) Using (i),(ii) and (iii) you have proven that for a given $\alpha^* > 0$, the solution of $(A^*A + \alpha^* I)u^* = A^*f$ is uniquely determined. Prove that u^* is also the solution of the unconstrained variational problem

$$\min_{u\in\mathcal{X}} \phi_{\alpha^*}(u) \quad \text{where} \quad \phi_{\alpha^*}(u) = \|f - Au\|^2_{\mathcal{Y}} + \alpha^*\|u\|^2_{\mathcal{X}}.$$

(c) We define the shrinkage operator as

$$\psi_\lambda(x) = \arg\min_{z\in\mathbb{R}^N} g_\lambda(z) \quad g_\lambda(z) = \frac{1}{2}\|z - x\|^2_2 + \lambda\|z\|_1 \tag{1}$$

where $g_\lambda : \mathbb{R}^N \to \mathbb{R}$. [*Hint: You can use the expression of the subdifferential of the absolute value without proving it.*]

(i) Prove that the components of the shrinkage operator are given by

$$[\psi_\lambda(x)]_i = \begin{cases} x_i - \lambda & \text{for} \quad x_i > \lambda, \\ 0 & \text{for} \quad -\lambda \leqslant x_i \leqslant \lambda, \\ x_i + \lambda & \text{for} \quad x_i < -\lambda. \end{cases} \tag{2}$$

(ii) Draw a plot of $[\psi_\lambda(x)]_i$.

END OF PAPER