UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 10 June, 2022 1:30 pm to 3:30 pm

PAPER 325

QUANTUM INFORMATION, FOUNDATIONS AND GRAVITY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Give equations defining the non-relativistic Ghirardi-Rimini-Weber (GRW) "spontaneous collapse" model for N distinguishable spin 0 particles, explaining carefully how it modifies the Schrödinger equation. Using a sketch model, explain how, with suitable parameter choices (which you should estimate), the GRW model predicts negligible deviations from the Schrödinger equation for an isolated microscopic system, but a swift collapse corresponding to a specific measurement outcome when the system undergoes a measurement-type interaction with a macroscopic apparatus.

Show that, if a single free particle of mass m undergoes a single GRW collapse, the expectation values of $\langle x \rangle$, $\langle p \rangle$ and $\langle x^2 \rangle$ do not immediately change. Show that the expectation value $\langle p^2 \rangle$ does immediately change. Explain what type of tests of the GRW model this suggests.

2

A quantum system S comprises two subsystems A and B, whose degrees of freedom are described by finite-dimensional Hilbert spaces H_A and H_B. Suppose that S is initially in a pure state $|\Psi\rangle_{AB}$. Explain what is meant by saying that $|\Psi\rangle_{AB}$ is *entangled*. Explain carefully how to define the reduced density matrices for each subsystem.

Suppose that two physically separated observers, Alice and Bob, control subsystems A and B respectively. Starting from the postulates of non-relativistic quantum mechanics, show that any measurement Bob carries out on subsystem B does not alter the reduced density matrix for subsystem A. Explain the implications for the relationship between quantum theory and special relativity.

Now suppose that Alice has a hypothetical device that gives, on request, a classical description of the quantum state of A, without altering that state. Explain how, if the device operates within the standard framework of non-relativistic quantum mechanics, Alice and Bob can use it to send superluminal signals.

Describe an alternative version of this device, defined in Minkowski space-time, that would not allow superluminal signalling. Explain carefully the physical assumptions underlying your definitions. Explain carefully why postulating the device does not lead to any logical inconsistency with those assumptions. Explain also why, given those assumptions, it is not possible to signal superluminally by any algorithm involving any combination of these devices together with unitary evolutions and measurements.

3

Bose et al. have proposed an experiment creating two separate quantum subsystems, 1 and 2, each of which is a spherical mass $m \sim 10^{-14}$kg in a superposition of two distinct position states, L and R. Suppose that these systems are allowed to fall for time t such that the R state of subsystem 1 and the L state of subsystem 2 are separated by $d_{RL} = d \sim 2 \times 10^{-4}$m, while the other three separations are much larger, so that their gravitational interactions are relatively negligible in the experiment. Explain carefully why, if all interactions other than gravity are also negligible, a standard quantum treatment of the evolution suggests that subsystems 1 and 2 should generally become entangled.

Explain what is meant by an *entanglement witness*. Show that the measure

$$W = |\langle \sigma_x^1 \otimes \sigma_z^2 \rangle + \langle \sigma_y^1 \otimes \sigma_y^2 \rangle|$$

discussed by Bose et al. is (up to multiplication and addition of constant factors) an entanglement witness. Assuming that the Pauli matrices are defined in the $|L\rangle, |R\rangle$ basis for each subsystem, so that for example $\sigma_z|L\rangle = |L\rangle, \sigma_z|R\rangle = -|R\rangle$, obtain a rough estimate (to the nearest integer) of the value of W after 10s of free fall in the given configuration.

[Newton's gravitational constant $G \sim 6.674 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$; Planck's constant $h \sim 6.626 \times 10^{-34}\text{kgm}^2\text{s}^{-1}$; $\hbar = \frac{h}{2\pi} \sim 1.054 \times 10^{-34}\text{kgm}^2\text{s}^{-1}$.]

4

Let $|\psi\rangle$ be an entangled pure state of two qubits (which we will take to represent spin-1/2 particles). Explain why $|\psi\rangle$ can be written in the form

$$|\psi\rangle = c_0 |\uparrow\rangle_1 |\downarrow\rangle_2 + c_1 |\downarrow\rangle_1 |\uparrow\rangle_2 \,,$$

where $\{|\uparrow\rangle_j, |\downarrow\rangle_j\}$ form an orthonormal basis for qubit j and the c_i are real and positive.

Let $\mathbf{a}, \mathbf{b}$ be any two vectors in $\mathbb{R}^3$. Show that the expected value of a measurement of the operator $\mathbf{a}.\boldsymbol{\sigma} \otimes \mathbf{b}.\boldsymbol{\sigma}$ in the state $|\psi\rangle$ is

$$E(\mathbf{a}, \mathbf{b}) = \langle \mathbf{a}.\boldsymbol{\sigma} \otimes \mathbf{b}.\boldsymbol{\sigma} \rangle_\psi = 2c_0 c_1 (a_x b_x + a_y b_y) - a_z b_z \,,$$

where the Pauli matrices are defined in the above-defined orthonormal basis for their respective qubit.

Consider the vectors $\mathbf{a} = (0,0,1), \mathbf{b} = (\sin\beta, 0, \cos\beta), \mathbf{a}' = (-1,0,0), \mathbf{b}' = (\sin\beta', 0, \cos\beta')$. Show that

$$|E(\mathbf{a}, \mathbf{b}) - E(\mathbf{a}, \mathbf{b}')| + |E(\mathbf{a}', \mathbf{b}) + E(\mathbf{a}', \mathbf{b}')| = |\cos\beta - \cos\beta'| + 2c_0 c_1 |\sin\beta + \sin\beta'| \,.$$

Hence show that there are pairs of measurement operators on the individual qubits whose correlations cannot be reproduced by any separable state.

Explain briefly the theoretical implications for witnessing entanglement in a BMV experiment, assuming that any relevant measurement is possible in practice and that decoherence and non-gravitational forces can be neglected.

END OF PAPER