UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 10 June, 2022 9:00 am to 12:00 pm

PAPER 324

QUANTUM COMPUTATION

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) In a laboratory we are able to perform computational basis measurements and the following set of unitaries: $\{H, S, CNOT, SWAP\}$ on any qubit or a pair of qubits. Show how we can implement the following one- and two-qubit measurements (while retaining the full post-measurement state):

(i) Z, X

(ii) $X \otimes X$, $Z \otimes X$

(b) Consider $P, Q \in \mathcal{P}_n$, where $\mathcal{P}_n$ is the Pauli group on n qubits and let $|\psi\rangle$ be an eigenstate of P with eigenvalue $\lambda_P \in \{\pm 1\}$. Let P and Q anti-commute.

(i) Show that measuring Q on $|\psi\rangle$ returns $\lambda_Q = \pm 1$, each with probability $\frac{1}{2}$.

(ii) Show that an operator $V(\lambda_P, \lambda_Q) = \frac{1}{\sqrt{2}}(\lambda_P P + \lambda_Q Q)$ is a unitary Clifford operation. Furthermore, describe the subspace of Q where $|\psi\rangle$ is mapped onto.

(c)

(i) Describe the steps involved in a Pauli-based Computation on input state $|\alpha\rangle$.

(ii) Let C be a non-adaptive Clifford circuit on $n+t$ qubit input state $|0\rangle^{\otimes n} \otimes |\phi\rangle$, where $|\phi\rangle$ is an arbitrary t-qubit state, followed by $Z_1, \ldots Z_n$ measurements where $Z_i = \mathbb{I}_1 \otimes \cdots \otimes \mathbb{I}_{i-1} \otimes Z \otimes \mathbb{I}_{i+1} \otimes \cdots \otimes \mathbb{I}_{n+t}$. Show that C can be weakly simulated by a non-adaptive Pauli-based Computation process $P_1, \ldots, P_s$ with final $Z_1, \ldots, Z_n$ basis measurement outputs, where $P_i \in \mathcal{P}_t$ and $s \leqslant t$.

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(i) Define a stabilizer state $|\psi\rangle$ and the stabilizer group G of $|\psi\rangle$.

(ii) Write down all Pauli operators that stabilize a) $|001\rangle$, b) $|\Phi^+\rangle \otimes |+\rangle$, where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

(iii) State the Gottesman-Knill Theorem. Write down the stabilizer tableau (ignoring the signs) for $|\Psi_{in}\rangle = |+\rangle^{\otimes 3}$. Consider a sequence of gates applied to the input state $|\Psi_{in}\rangle$ that results in $|\Psi_{out}\rangle = CNOT_{13}(H_2 \otimes H_3)CNOT_{21}|\Psi_{in}\rangle$, where the subscript indicates the qubit it acts on, and $CNOT_{ij}$ has i-th qubit as control and j-th as target. Compute the stabilizer tableau (ignoring the signs) for $|\Psi_{out}\rangle$.

(iv) Show that $P = P_1 \otimes \cdots \otimes P_n$, and $Q = Q_1 \otimes \cdots \otimes Q_n$ where $Q_i, P_i \in \{\mathbb{I}, X, Y, Z\}$ either commute or anti-commute. When P and Q anticommute, determine the subspace that is stabilized by both of them.

(v) Define the stabilizer subspace $V_G = \{|\psi\rangle : P|\psi\rangle = |\psi\rangle, \forall P \in \mathrm{G}\}$, where G is a commuting subgroup of the Pauli group. Let $\Pi_G = \frac{1}{|G|}\sum_{P \in G} P$. Show that Π_G is a projector onto V_G. Consider the generators of G: $\langle g_1, \ldots, g_l\rangle$. Show that Π_G can be expressed as $\Pi_G = \prod_{i=1}^{l} \frac{1}{2}(\mathbb{I} + g_i)$.

3 (a) Let H be an n-qubit 2-local Hamiltonian with all of its $m = poly(n)$ terms commuting. Suppose we can produce a state $|\Psi\rangle$ which is an eigenstate of H with real eigenvalue $\lambda \in (0, 1)$.

(i) Describe an algorithm to implement $U(t) = e^{-iHt}$ with accuracy $\epsilon \in (0, 1)$ on any given input state $|\alpha\rangle$ for $t > 0$, clearly stating any results that you use. Show that $|\Psi\rangle$ is an eigenstate of $U(t)$ and find its eigenvalue.

(ii) Assume $U(t)$ can be implemented exactly and all functions of its eigenvalues and their values mod 2π can be represented using at most N binary digits. Using the Phase Estimation algorithm show how to determine λt mod 2π (ignoring the issues of accuracy).

(iii) Consider a 2-local Hamiltonian $H = 5X \otimes X \otimes \mathbb{I} - 4Z \otimes \mathbb{I} \otimes Z$. What is the smallest value of k for which H^4 is k-local?

(b) Consider an undirected graph $G = (V, E)$ on $|V| = n$ vertices. Each edge has weight $w_{i,j} = w_{j,i}$, $w_{i,j} = 1$ for $(i, j) \in E$, $w_{i,j} = 0$ if $(i, j) \notin E$. A cut in G is a partition of the set $V = S_1 \cup S_2$ into two disjoint subsets S_1, S_2. With each vertex $v_i \in V$ we associate a variable $x_i = 1$ if $x_i \in S_1$, and $x_i = 0$ if $x_i \in S_2$. The cost of the cut is the sum of weights of edges that connect vertices in the two different subsets, crossing the cut. Define the cost function $C(x_1, \dots, x_n) = \sum_{i,j=1}^{|V|} w_{ij} x_i (1 - x_j)$.

(i) Consider an undirected graph $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4, v_5\}$, $E = \{(1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$. Find a partition of V into two subsets that maximizes $C(x_1, \dots x_5)$.

(ii) Consider a Hamiltonian $H = -C(\hat{x}_1, \dots, \hat{x}_5)$, where x_i is replaced by $\hat{x}_i = \frac{1}{2}(\mathbb{I} - Z_i)$ and 1 is replaced by $\mathbb{I}$. Express H in terms of qubit projectors $P_0 = |0\rangle\langle 0|, P_1 = |1\rangle\langle 1|$. Assume that the smallest eigenvalue of H is bounded as $|\lambda_{min}| \leqslant 4$. Using (i) or otherwise, find the ground state $|\psi\rangle \in (\mathbb{C})^{\otimes 5}$ of H.

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(a)

(i) Define the Quantum Fourier Transform over a finite abelian group G.

(ii) Recall that an n-qubit state $|\psi\rangle$ is a product state if it can be represented as $|\alpha_1\rangle \otimes \cdots \otimes |\alpha_n\rangle$, where each $|\alpha_i\rangle$ is a 1-qubit state. Consider a state $|\phi\rangle = \frac{1}{T} \sum_{y=0}^{2^n-1} z^y |y\rangle$, where $z \in \mathbb{C}, z^{2^n} = 1$, T is a normalization constant. Show that $|\phi\rangle$ is a product state.

(b)

(i) State the Hidden Subgroup Problem (HSP).

(ii) Suppose we are given a quantum oracle for a function $f : \{0,1\}^n \to \{0,1\}^n$ such that $f(x) = f(y)$ if and only if $x \oplus y = t$, $x, y \in \{0,1\}^n$,$t \in \{0^n, p\}$, and $p \in \{0,1\}^n$ is fixed. The problem is to determine p. Show that this problem can be reduced to the HSP.

(iii) Describe a quantum algorithm that solves the HSP in (ii), clearly stating any relevant theorems that you use.

END OF PAPER