UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2022 1:30 pm to 4:30 pm

PAPER 323

QUANTUM INFORMATION THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $U \sim p(u)$, $u \in J$, where J is a finite alphabet, denote an i.i.d. classical information source.

[For this question you can assume that for all $\varepsilon > 0$, $\mathbb{P}[T_\varepsilon^{(n)}] \to 1$ as $n \to \infty$, where $T_\varepsilon^{(n)}$ is the ε-typical set as defined in the lecture. You should not assume any other result from the lecture for this question.]

(a) Explain what is meant by a *reliable compression-decompression scheme* of rate R for the source U. Show that if $R > H(U)$ such a scheme exists.

(b) Let a memoryless quantum information source be given by the density matrix π, acting on a Hilbert space $\mathcal{H}$. Explain what is meant by a *reliable quantum compression-decompression scheme* of rate R for the source π. Show that if $R > S(\pi)$ such a scheme exists, where $S(\pi)$ is the von Neumann entropy of π.

(c) Define the sets of sequences $C_n(r) := \{ (u_1, \ldots, u_n) \in J^n \mid p(u_1, \ldots u_n) \geqslant 2^{-nr} \}$. Show that, instead of using the sets of typical sequences, the result of a) can also be established using the $C_n(r)$ for a suitably chosen r.

2

Let Λ be a linear map $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ denotes a finite-dimensional Hilbert space.

(a) State what is meant by Λ being *completely positive*. Define the *Choi-matrix*, $J(\Lambda)$, of Λ. Show necessary and sufficient conditions on the Choi matrix for Λ to be completely positive. [Clearly state any result from the lecture you are using.]

(b) Show that if Λ can be written as

$$\Lambda(\rho) = \sum_k A_k \rho A_k^\dagger \tag{1}$$

then it is completely positive, where A_k are linear operators acting on the Hilbert space $\mathcal{H}$. Show a necessary and sufficient condition on the operators A_k for Λ to be trace preserving.

(c) Let $\{|y_i\rangle\}_{i=1}^d$ be an orthonormal basis of $\mathcal{H}$. Show that the channel

$$\mathcal{Y} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \qquad \mathcal{Y}(\rho) = \sum_i |y_i\rangle\langle y_i| \langle y_i|\rho|y_i\rangle \tag{2}$$

is trace preserving and completely positive. Explain why this channel can be called a *measure-and-prepare channel*. Show that $S(\mathcal{Y}(\rho))$ is the Shannon entropy of the measurement outcome probabilities. Show that

$$D(\rho\|\mathcal{Y}(\rho)) = S(\mathcal{Y}(\rho)) - S(\rho)\,. \tag{3}$$

(d) Let $\{|z_i\rangle\}_{i=1}^d$ be another orthonormal basis of $\mathcal{H}$ and define similarly

$$\mathcal{Z} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \qquad \mathcal{Z}(\rho) = \sum_i |z_i\rangle\langle z_i| \langle z_i|\rho|z_i\rangle\ . \tag{4}$$

Evaluate the expression $D(\mathcal{Z}(\rho)\|\mathcal{Z}(\mathcal{Y}(\rho)))$ to show that

$$S(\mathcal{Y}(\rho)) + S(\mathcal{Z}(\rho)) \geqslant -\log\left(\max_{i,j} |\langle y_i|z_j\rangle|^2\right) + S(\rho)\,. \tag{5}$$

3

(a) Let $\{p_x, \rho_B^x\}$ and $\{q_x, \sigma_B^x\}$ be two ensembles of quantum states with the same (finite) number of elements. We define the states

$$\rho_{XB} = \sum_x p_x \, |x\rangle\langle x|_X \otimes \rho_B^x \tag{1}$$

$$\sigma_{XB} = \sum_x q_x \, |x\rangle\langle x|_X \otimes \sigma_B^x \, . \tag{2}$$

Show that

$$D(\rho_{XB} \| \sigma_{XB}) = \sum_x p_x D(\rho_B^x \| \sigma_B^x) + D(\underline{p} \| \underline{q}) \, , \tag{3}$$

where $\underline{p} = \{p_x\}$ and $\underline{q} = \{q_x\}$, Conclude that

$$S(\rho_{XB}) = \sum_x p_x S(\rho_B^x) + H(\underline{p}) \, , \tag{4}$$

where $H(\underline{p})$ denotes the Shannon entropy of the probability distribution $\underline{p}$. [You can use the fact that the logarithm of a block-diagonal matrix is the block-diagonal matrix consisting of the logarithms of the individual blocks (formally: $\log(A \oplus B) = \log(A) \oplus \log(B)$).]

(b) State what is meant by (*i*) the *data-processing inequality* for the quantum relative entropy, (*ii*) *joint convexity* of the quantum relative entropy, and (*iii*) the *strong subadditivity* inequality of the von Neumann entropy. Show that the data-processing inequality implies both strong subadditivity and joint convexity. [You may assume that the partial trace is a CPTP map.]

(c) State and prove the Holevo bound.
[Hint: If you wish, you can use the fact that for any POVM $\{\Lambda_x\}_{x=1}^m$ the channel

$$\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathbb{C}^m) \qquad \rho \mapsto \sum_x |x\rangle\langle x| \, \mathrm{Tr}(\rho \Lambda_x) \tag{5}$$

is CPTP, together with the data processing inequality.]

(d) Let Λ be a quantum channel. Consider the quantity

$$\chi^*(\Lambda) := \max_{\{p_x, \rho_x\}} \chi(\{p_x, \Lambda(\rho_x)\}) \tag{6}$$

where χ is the Holevo χ quantity. Explain the operational meaning of this quantity and show that the maximization can be restricted to ensembles of pure states [clearly state any result from the lecture you are using].

4 Let $\mathcal{H}$ be a Hilbert space and $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ be two density matrices.

(a) Show that the trace distance is monotonous under quantum channels, i.e. for any linear CPTP map

$$\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \tag{1}$$

it holds that

$$\frac{1}{2}\|\Lambda(\rho) - \Lambda(\sigma)\|_1 \leqslant \frac{1}{2}\|\rho - \sigma\|_1 \,. \tag{2}$$

[Clearly state any result from the lecture you are using and justify your steps.]

(b) Show that there exists a positive operator $0 \leqslant P \leqslant \mathbb{1}$ such that

$$\|\rho - \sigma\|_1 = |\mathrm{Tr}(P(\rho - \sigma))| + |\mathrm{Tr}((\mathbb{1} - P)(\rho - \sigma))| \,. \tag{3}$$

(c) Show that for any POVM $\{\Lambda_k\}_{k=1}^m$, and any orthonormal basis $\{|k\rangle\}_{k=1}^m$ of $\mathbb{C}^m$, the channel

$$\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathbb{C}^m) \qquad \Lambda(\rho) = \sum_{k=1}^{m} |k\rangle\langle k| \, \mathrm{Tr}(\Lambda_k \rho) \tag{4}$$

is CPTP. [*Hint: Find a set of Kraus operators to show complete positivity.*]

(d) Show that

$$\|\rho - \sigma\|_1 = \max_{\{\Lambda_k\} \text{ POVM}} \sum_k |\mathrm{Tr}(\Lambda_k \rho) - \mathrm{Tr}(\Lambda_k \sigma)| \tag{5}$$

where the maximization is over all POVMs $\{\Lambda_k\}$ (with arbitrarily many elements).

(e) The fidelity of two quantum states is defined as

$$F(\rho, \sigma) = \mathrm{Tr}\left[\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}\right] . \tag{6}$$

It can also be expressed as an minimization over all POVMs:

$$F(\rho, \sigma) = \min_{\{\Lambda_k\} \text{ POVM}} \sum_k \sqrt{\mathrm{Tr}(\Lambda_k \rho)\,\mathrm{Tr}(\Lambda_k \sigma)} \,. \tag{7}$$

Use this to show that

$$\frac{1}{2}\|\rho - \sigma\|_1 \geqslant 1 - F(\rho, \sigma) \tag{8}$$

[*Hint: Use that* $|p - q| \geqslant (\sqrt{p} - \sqrt{q})^2$ *for all* $p, q \geqslant 0$.]

5

(a) Show that for any pure bipartite state $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$,

$$S(\rho_A) = S(\rho_B)\,. \tag{1}$$

Consider a channel $\Lambda : \mathcal{B}(\mathcal{H}_Q) \to \mathcal{B}(\mathcal{H}_Q)$ with associated Stinespring isometry

$$U : \mathcal{H}_Q \to \mathcal{H}_Q \otimes \mathcal{H}_E \qquad \Lambda(\rho) = \mathrm{Tr}_E(U\rho U^\dagger)\,. \tag{2}$$

Let $\rho \in \mathcal{D}(H_Q)$ be an arbitrary input state and $|\phi_{RQ}\rangle$ be a purification of ρ. We define the pure state

$$|\psi_{RQE}\rangle \coloneqq (\mathbb{1}_R \otimes U)\,|\phi_{RQ}\rangle \tag{3}$$

and the associated density matrix

$$\sigma_{RQE} \coloneqq |\psi_{RQE}\rangle\langle\psi_{RQE}|\;. \tag{4}$$

(b) State the definition of $I_c(\Lambda, \rho)$, the *coherent information of a channel with respect to the input state* ρ.

(c) Show that there always exists an input state ρ such that

$$I_c(\Lambda, \rho) = 0\,. \tag{5}$$

(d) Prove the following relation

$$I_c(\Lambda, \rho) = \frac{1}{2}(I(R:Q) - I(R:E)) \tag{6}$$

where the mutual informations on the right-hand side are with respect to the state σ_{RQE}.

(e) The channel Λ is called *anti-degradable* if there exists a linear CPTP map $\mathcal{N} : \mathcal{B}(\mathcal{H}_E) \to \mathcal{B}(\mathcal{H}_Q)$ such that for all input states $\rho \in \mathcal{D}(H_Q)$ with a purification $|\phi_{RQ}\rangle$ and associated state σ_{RQE} (as defined through (3) and (4)) it holds that $(\mathrm{id}_R \otimes \mathcal{N})(\sigma_{RE}) = \sigma_{RQ}$. Show that if Λ is anti-degradable then for any $\rho \in \mathcal{D}(H_Q)$

$$I_c(\Lambda, \rho) \leqslant 0 \tag{7}$$

and hence show that then

$$Q^{(1)}(\Lambda) = \max_\rho I_c(\Lambda, \rho) = 0\,. \tag{8}$$

[QUESTION CONTINUES ON THE NEXT PAGE]

(f) In its most general form, the no-cloning theorem when applied to pure states says that there does not exist any quantum channel $\mathcal{M}$ such that for all pure states ρ

$$\mathcal{M}(\rho) = \rho \otimes \rho\,. \tag{1}$$

We say that a quantum channel Λ can be used to transmit quantum information perfectly over one use if there exists an encoding map $\mathcal{E}$, and a decoding map $\mathcal{D}$, such that $\mathcal{D} \circ \Lambda \circ \mathcal{E} = \mathrm{id}$, where id denotes the identity channel.

Prove that if an anti-degradable channel could be used to transmit quantum information perfectly over one use, then the above statement of the no-cloning theorem would be violated.

Hint: The following two facts might be helpful:

1. *If* $\Lambda_1 : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_{A'})$ *and* $\Lambda_2 : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_{B'})$ *are two quantum channels, then* $\mathrm{Tr}_{B'}((\Lambda_1 \otimes \Lambda_2)(\omega_{AB})) = \Lambda_1(\omega_A)$ *for any bipartite density matrix* ω_{AB}*.*
2. *For any bipartite density matrix* ω_{AB}*, if* $\omega_A = \mathrm{Tr}_B(\omega_{AB})$ *is pure, then* $\omega_{AB} = \omega_A \otimes \omega_B$*.*

END OF PAPER