UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 8 June, 2022 1:30 pm to 3:30 pm

PAPER 321

DYNAMICS OF ASTROPHYSICAL DISCS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The vertical structure of self-gravitating discs

(a) In a Keplerian disc the Toomre parameter is $Q = \Omega c_s/(\pi G\Sigma)$, where c_s is the sound speed and the other symbols take their usual meanings. Describe the different physical processes competing over the gravitational stability of the disc and how they are represented in the Toomre parameter. If gravitational instability leads to a turbulent state, why might you expect this state to settle on $Q \sim 1$?

(b) The equations determining the vertical structure of a thin Keplerian self-gravitating alpha disc are

$$\frac{dP}{dz} = -\rho\Omega^2 z - \rho\frac{d\Phi_d}{dz}, \qquad \frac{d^2\Phi_d}{dz^2} = 4\pi G\rho,$$
$$\frac{dF}{dz} = \frac{9}{4}\alpha P\Omega, \qquad \frac{dT}{dz} = -\frac{3}{16}\frac{\kappa\rho}{\sigma T^3}F,$$

where Φ_d is the disc's gravitational potential, κ is the opacity, and the other symbols take their usual meanings. Explain what each equation represents. Adopt the scalings $P/\rho \sim c_s^2$ and $\Sigma \sim \rho H$, where H is the disc's semi-thickness, and show that if all the terms in vertical hydrostatic equilibrium are of the same order then $Q \sim 1$.

(i) Assume that the disc is composed of perfect gas, i.e. $P = (k\rho T)/(\mu_m m_p)$, and that κ is a constant.

By taking an order of magnitude approach to the above equations and enforcing $Q \sim 1$, show that α is not a free parameter but scales as

$$\alpha \sim \frac{\sigma}{\kappa}\left(\frac{\mu_m m_p \Sigma}{k}\right)^4 G^6 \Omega^{-7},$$

and hence $\bar{\nu} \propto r^{15}\Sigma^6$, where $\bar{\nu}$ is the mean turbulent viscosity.

(ii) Suppose the gas can be described by a polytropic equation of state, i.e. $P = K\rho^{1+1/n}$, where K is a constant and n is the polytropic index. Set $c_s^2 = dP/d\rho$, and denote by c_0 and ρ_0 the midplane sound speed and midplane density.

The dimensionless pseudo-enthalpy w is defined so that $dw = (c_0^2\rho)^{-1}dP$. Show that $\rho = \rho_0 n^{-n} w^n$, and that w satisfies

$$\frac{d^2w}{d\zeta^2} + \frac{4}{n^n Q_0}w^n = -1,$$

where $\zeta = (\Omega/c_0)z$, $Q_0 = \Omega c_0/(\pi G\Sigma)$, and $\Sigma = \rho_0(c_0/\Omega)$, with boundary conditions $w(0) = n$ and $w'(0) = 0$. For the special case $n = 1$, solve the differential equation and find H as a function of Q_0, K, ρ_0, and Ω.

2 Spreading of a narrow planetary ring

The equations of a razor-thin compressible planetary ring in the shearing sheet are

$$\frac{\partial \Sigma}{\partial t} + \mathbf{u}\cdot\nabla\Sigma = -\Sigma\nabla\cdot\mathbf{u},$$
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -2\Omega\mathbf{e}_z\times\mathbf{u} - \nabla\Phi_t - \frac{1}{\Sigma}\nabla P + \frac{1}{\Sigma}\nabla\cdot\mathbf{T},$$

where $\Phi_t = -(3/2)\Omega^2 x^2$ and the viscous stress tensor $\mathbf{T}$ is given by

$$T_{ij} = \Sigma\nu\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}(\nabla\cdot\mathbf{u})\delta_{ij}\right].$$

(a) Suppose $\Sigma = \Sigma(x,t)$, $u_x = u_x(x,t)$, $P = P(x,t)$, and (approximately) $u_y = -(3/2)\Omega x$. By first finding an expression for u_x in terms of Σ, derive the diffusion equation in the shearing sheet:

$$\frac{\partial \Sigma}{\partial t} = 3\frac{\partial^2(\nu\Sigma)}{\partial x^2}. \qquad (\dagger)$$

Explain what kind of evolution this equation approximately describes and what kinds of behaviour it cannot describe.

(b) Assume that $\nu = \nu(\Sigma)$. Starting from (†), conduct a linear perturbation analysis of the homogeneous state $\Sigma = \Sigma_0$ and show that the planetary ring is unstable to the viscous instability when $d(\nu\Sigma)/d\Sigma < 0$. In what way will the ring evolve under the action of this instability?

(c) Now assume that $\nu = A\Sigma^2$, where A is a dimensional constant, and adopt the following form for Σ, describing a spreading narrow ring,

$$\Sigma = \begin{cases} \sigma(t)\sqrt{1 - x^2/w(t)^2}, & |x| < w, \\ 0, & |x| > w, \end{cases}$$

where σ and w are functions of time, yet to be determined.

Find the form of u_x inside the ring. Hence derive the evolution equation $\dot{w} = 9A\sigma^2/w$.

By direct substitution into (†), show that $\dot{\sigma} = -\sigma\dot{w}/w$, and hence that σw is a constant. Demonstrate that conservation of total mass in the ring is ensured if σw is a constant.

Given that $\sigma = \sigma_0$ and $w = w_0$ at $t = 0$, find expressions for w and σ in the form $w = w_0 f(t)$, $\sigma = \sigma_0 g(t)$, where f and g are functions to be determined. Show that at large times $w \propto t^{1/4}$.

3 The stratified incompressible shearing sheet

The incompressible shearing sheet can be extended so as to account for vertical buoyancy. In a Keplerian disc, the governing equations are then

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho_0}\nabla P - 2\Omega\mathbf{e}_z\times\mathbf{u} + 3\Omega^2 x\mathbf{e}_x - N^2\theta\mathbf{e}_z,$$
$$\partial_t\theta + \mathbf{u}\cdot\nabla\theta = u_z, \qquad \nabla\cdot\mathbf{u} = 0,$$

where θ is the buoyancy variable (with units of length), N^2 is the squared buoyancy frequency, and other symbols take their usual meanings. Both N^2 and ρ_0 are constants.

(a) Briefly describe the rationale and main approximations of the (standard) incompressible shearing sheet. Which kinds of flow can it represent?

(b) The total specific energy is $E = \frac{1}{2}(|\mathbf{u}|^2 + N^2\theta^2)$. Show that E obeys the conservation law $\partial_t E + \nabla\cdot\mathbf{F} = 0$, where $\mathbf{F}$ is an energy flux, the form of which you need to find.

[*You may need the vector identity:* $\frac{1}{2}\nabla|\mathbf{u}|^2 = \mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{u})$.]

(c) Verify that $\mathbf{u} = \mathbf{u}_0 = -(3/2)\Omega x\mathbf{e}_y$, $P = P_0$ (a constant), and $\theta = 0$ is a solution to the governing equations.

Perturb this equilibrium with disturbances $\propto e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$, where $\mathbf{k} = \mathbf{e}_x k_x + \mathbf{e}_z k_z$, and ω is a wave frequency. Write down the perturbation equations, showing that all the nonlinear terms are precisely zero.

Derive the dispersion relation

$$\omega^2 = \frac{k_z^2}{k^2}\Omega^2 + \frac{k_x^2}{k^2}N^2,$$

where $k^2 = k_x^2 + k_z^2$.

The group velocity $\mathbf{c}$ is defined so that $c_i = \partial\omega/\partial k_i$. If $N^2 = 0$, show that $\mathbf{c} = \pm k^{-3}\left[\mathbf{k}\times(\boldsymbol{\Omega}\times\mathbf{k})\right]$. What is $\mathbf{c}$'s relative orientation to the phase velocity? How might a packet of inertial waves propagate?

If $N^2 > 0$, consider the two limits $k_x/k_z \to 0$ and $k_z/k_x \to 0$. Which kinds of motion and what physical processes dominate in each limit and why? How might you characterise oscillations with intermediate k_x/k_z?

If $N^2 < 0$, derive an instability criterion for the onset of convection. Why are only some modes unstable?

END OF PAPER