UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 13 June, 2022 1:30 pm to 3:30 pm

PAPER 320

MODERN STELLAR DYNAMICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider an oblate spheroidal galaxy with axis ratio $q \leqslant 1$, whose minor axis is directed at an inclination angle i to the line of sight.

(a) Derive the expression for the apparent (projected) axis ratio Q.

(b) Under the natural assumption that galaxies are randomly oriented in the 3d space (i.e., the direction of the minor axis is uniformly distributed on a unit sphere), what is the probability distribution of inclination angles?

Derive the probability distribution $\mathcal{P}(Q)$ for an ensemble of randomly oriented galaxies in two cases:
(c) assuming that they all share the same intrinsic axis ratio q, or
(d) assuming that they follow a certain distribution of intrinsic axis ratios $\mathcal{P}(q)$.

(e) If $\mathcal{P}(q)$ is a uniform distribution, what can be said about $\mathcal{P}(Q)$ – is it uniform, or skewed towards lower or higher Q?

(f) Determine $\mathcal{P}(Q)$ for the case $\mathcal{P}(q) = \frac{4}{\pi}\sqrt{1-q^2}$.

(g) If $\mathcal{P}(Q)$ is uniform, what can be deduced about $\mathcal{P}(q)$?

(h) Can $\mathcal{P}(Q)$ be skewed towards lower Q, and if yes, under what conditions (not necessarily compatible with the above derivations)?

2 The spherical isochrone model with mass M and scale radius b has the potential

$$\Phi(r) = -\frac{G\,M}{b + \sqrt{b^2 + r^2}}.$$

(a) Determine the asymptotic behaviour of its density profile at small and large radii.

(b) Compute the period of radial oscillations and show that it does not depend on L, only on E (hence the name "isochrone").

(c) Compute the ratio of radial and azimuthal periods, explain its asymptotic behaviour at small and large radii.

(d) Determine the radial action J_r and show that the Hamiltonian of the system can be expressed in terms of actions as

$$H(J_r, L) = -\frac{2(G\,M)^2}{\left(2J_r + L + \sqrt{L^2 + 4\,G\,M\,b}\right)^2}.$$

In fact, this is almost the only non-trivial example of a fully analytic transformation between phase-space and action–angle variables!

Hint: it is convenient to introduce a scaled radial variable $s \equiv -G\,M\big/\big[\Phi(r)\,b\big]$.

Useful expressions ($c \leqslant a \leqslant b$):

$$\int_a^b \mathrm{d}x\, \frac{x - c}{\sqrt{(x-a)(b-x)}} = \pi\left[\frac{a+b}{2} - c\right],$$

$$\int_a^b \mathrm{d}x\, \frac{1}{\sqrt{(x-a)(b-x)}\;(x-c)} = \frac{\pi}{\sqrt{(a-c)(b-c)}},$$

$$\int_a^b \mathrm{d}x\, \frac{\sqrt{(x-a)(b-x)}}{x-c} = \pi\left[\frac{a+b}{2} - c - \sqrt{(a-c)(b-c)}\right].$$

3

Consider a spherical stellar system with the following distribution function:

$$f(E, L) = L^{p-2}\, g(E), \quad 0 < p \leqslant 2.$$

(a) Show that the velocity dispersions in this system satisfy $\sigma_\theta^2 = \sigma_\phi^2 = (p/2)\,\sigma_r^2$ for any $g(E)$.

Consider now a spherical model with a gravitational potential

$$\Phi(r) = -\frac{G\,M}{(r^p + a^p)^{1/p}}, \quad \text{where } M \text{ is the total mass and } a \text{ is the scale radius.}$$

(b) Write down the corresponding density profile from the Poisson equation, and show that this model can be self-consistently generated by a distribution function

$$f(E, L) = A\, L^{p-2}\, (-E)^{(3p+1)/2}$$

for a suitable choice of A; determine its value.

(c) Compute the velocity dispersions in this model and show that they are proportional to the value of the gravitational potential. Specifically, demonstrate that the kinetic and potential energy satisfy the virial relation *at any radius*. Is this a generic property of stellar systems?

You may use the definition of the Beta function:

$$\int_0^1 \mathrm{d}x\; x^{a-1}\, (1-x)^{b-1} = \mathrm{B}(a, b) \equiv \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)},$$

and recall that $\Gamma(x+1) = x\,\Gamma(x)$.

END OF PAPER