UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2022 9:00 am to 11:00 am

PAPER 319

UNBOUNDED OPERATORS AND SEMIGROUPS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** parts of the question.
There is **ONE** question in total.

In doing a given part of the question you may use assertions in preceding parts even if you did not complete that part.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In this question $\big(H, \|\cdot\|\big)$ is a Banach space and $\{U(t)\}_{t\geqslant 0}$ is a C_0 semigroup of operators in $B(H)$ which verifies $\|U(t)u\| \leqslant Me^{\omega t}\|u\|\,\forall u \in H$. Let $\mathcal{G}(M,\omega)$ be the set of generators of such semigroups.

(a) Define the generator $A \in \mathcal{G}(M,\omega)$ of the semigroup U and write down, without justification, an element of $\mathrm{Dom}\,(A)$. Show that A is closed and its resolvent set $\rho(A) \supset \{z : \mathrm{Re}\,(z) > \omega\}$.

(b) Prove that $n!(z-A)^{-(n+1)}\,u = \int_0^\infty t^n e^{-tz} U(t)\,u\,dt$ for $n \in \mathbb{N}\cup\{0\}, \mathrm{Re}\,(z) > \omega$, and that for $\lambda > \omega$

$$\|(\lambda - A)^{-(n+1)}u\| \leqslant M(\lambda-\omega)^{-n+1}\|u\| \quad \text{for all } u \in H\,.$$

(c) Show that the domain $\mathrm{Dom}\,(A)$ endowed with the norm $\|u\|_Y = \|u\| + \|Au\|$ is a Banach space Y.

(d) Show that the subspace $Y = \mathrm{Dom}\,(A)$ is invariant under $\{U(t)\}_{t\geqslant 0}$, and that $\tilde{U}(t) = U(t)\big|_Y$, the restriction of the semigroup $\{U(t)\}_{t\geqslant 0}$ to Y, is itself a C_0 semigroup on Y whose generator is the restriction of A to

$$\mathrm{Dom}\,(A^2) = \{u \in \mathrm{Dom}\,(A) : Au \in \mathrm{Dom}\,(A)\}\,.$$

Obtain a bound for $\|\tilde{U}(t)y\|_Y$ in terms of M, ω, t and $\|y\|_Y$.

(e) Show that $\hat{U}(t) = e^{-\omega t}U(t)$ defines a C_0 semigroup of operators in $B(H)$ which verifies $\|\hat{U}(t)u\| \leqslant M\|u\|\,\forall u \in H$ and that $\hat{A} = A - \omega$ is its generator (so that $\hat{A} \in \mathcal{G}(M,0)$.) State the Hille-Yosida theorem for the case of generators in $\mathcal{G}(M,0)$.

(f) Explain the notion of solution operator for an evolution equation $\partial_t u = A(t)u$. State a theorem on the existence of a solution operator $\{U(t,s)\}_{0\leqslant s\leqslant t\leqslant T}$ on finite time intervals $[0,T]$ for the equation $\partial_t u = A(t)u$, when $A(t) \in \mathcal{G}(M,0)$ for each t, giving conditions on $\{A(t)\}_{0\leqslant t\leqslant T}$ and the resulting properties of $\{U(t,s)\}_{0\leqslant s\leqslant t\leqslant T}$. Write down operators which are a finite product

$$\prod_{j=1}^{N} \exp[s_j A(t_j)] \qquad \text{(for appropriate } \{s_j\} \text{ and } \{t_j\})$$

which converge to $U(t,s)$ as $N \to \infty$, state the type of convergence, and prove that $U(t,s)u$ is differentiable with respect to s for $u \in \mathcal{D}$.

(g) Given $\phi \in C^1(\mathbb{R};\mathbb{R})$, consider the operator $A(t) = \partial_x^3 + \phi(t)\partial_x$, regarded as an unbounded operator on $L^2(\mathbb{R}, dx)$ with domain

$$\mathrm{Dom}\,(A) = H^3(\mathbb{R}) = \{u \in L^2 : (1+\xi^2)^{3/2}\hat{u}(\xi) \in L^2(\mathbb{R}, d\xi)\}\,,$$

where $\hat{u}$ is the Fourier transform. Show that this satisfies the conditions in (f), and deduce the existence of a solution operator $\{U(t,s)\}_{0\leqslant s\leqslant t\leqslant T}$ for the equation $\partial_t u = A(t)u$ for any positive $T < \infty$.

[QUESTION CONTINUES ON THE NEXT PAGE]

(h) Continuing from the previous part, prove that given $u_0 \in L^2(\mathbb{R})$ there is a unique $u \in C([0,T]; L^2(\mathbb{R}))$ such that

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s) f(s, u(s)) ds \,,$$

for $f \in C([0,T] \times L^2(\mathbb{R}); L^2(\mathbb{R}))$ obeying, for all $t \in [0,T]$ and u, v in $L^2(\mathbb{R})$,

$$\|f(t,u)\| \leqslant C \quad \text{and} \quad \|f(t,u) - f(t,v)\|_{L^2} \leqslant L \|u - v\|_{L^2} \,,$$

for some positive numbers C, L.

END OF PAPER