UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2022 9:00 am to 12:00 pm

PAPER 317

STRUCTURE AND EVOLUTION OF STARS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a spherically symmetric star in radiative equilibrium. The stellar material is a perfect gas with gas pressure P_{g} and radiation pressure P_{rad}. The total pressure is $P = P_{\mathrm{g}} + P_{\mathrm{rad}}$ and $P = P(r)$ where r is the radius. The star is chemically homogeneous. The opacity is given by Kramer's formula.

Consider the above model under the assumption that the rate of energy release per unit mass ϵ is described by $\epsilon = \mathrm{const}$, that is there is a uniform distribution of sources of energy.

(i) Write down the equations of equilibrium. Show that in the above model $\epsilon = M^{\mathrm{a}}\, L^{\mathrm{b}}$ where M and L are the total mass and luminosity of the star. Determine the values of a and b.

(ii) Introduce new variables $y = \beta/(1-\beta)$ with $\beta = P_{\mathrm{g}}/P$ and $x = \dfrac{4\,\pi\, c\, G\, M}{\kappa_0\, L}\, \dfrac{k}{\mu\, H}\, \dfrac{3}{a}\, T^{1/2}$, where κ_0 is a constant in the Kramer's formula for the opacity, H is the mass of a hydrogen atom and all other constants have their usual meaning. Derive, using the structure equations in (i)

$$\frac{1}{8}\, x\, y\, \frac{\mathrm{d}\, y}{\mathrm{d}x} = x - y\,(y+1)\,. \tag{1}$$

(iii) A solution to equation (1) may be found by a particular expansion of y. You do not need to worry how to find the solution but simply assume that it is given approximately by

$$x = \frac{32}{31} y\, (\frac{32}{31}\, y + 1).$$

Show that the star is a polytrope and find in terms of β the effective polytropic index $n = n(\beta)$ relating pressure and density in a standard form $P \propto \varrho^{1+1/n}$. Find n for $\beta = 0$ and $\beta = 1$.

2

Consider the point-source model of a star of mass M, radius R. This means that we assume that the source of energy is just at the center of the star, thus $L(r) = \text{const.} = L$. The star is spherically symmetric. It is in radiative equilibrium. The stellar material is a perfect gas with gas pressure P_{g} and the radiation pressure is negligible. The total pressure is therefore $P = P_{\mathrm{g}}$ and $P = P(r)$ where r is the radius. The star is chemically homogeneous. The opacity is given by Kramer's law. Assume that at the surface of the star $T = 0$ and $P = 0$.

(i) Assume that the star is also so centrally condensed that its gravitational field can be approximated by placing a point mass at the centre. This means that the outside layers have mass so small that variations of mass m_{r} can be neglected though the density ϱ of matter is not negligible.

Write down the equations describing this model of a star assuming that the whole star is in local thermodynamic equilibrium.

Find the pressure P and density ϱ as functions of the temperature T.

Find the temperature profile $T(r)$ for this star.

(ii) Assume now that the star has an isothermal electron-degenerate non-relativistic core which contains essentially all mass of the star. The pressure given by electron-degenerate non-relativistic matter is $\tilde{K}\,(\varrho/\mu_{\mathrm{e}})^{5/3}$, where μ_{e} is the mean molecular weight of the electrons and $\tilde{K}$ is a constant. The core has luminosity L. Outside the core, the structure of the star is as determined by relations $P(T)$ and $\varrho(T)$ which were worked out in (i). Assume that all physical parameters are continuous between the core and the outer layers and the core temperature is T_{c}. Find L/M for this star in terms of T_{c}.

3

Consider a mixture of a perfect homogenous gas and black body radiation. The total pressure is given by $P = P_{\mathrm{g}} + P_{\mathrm{rad}}$ where P_{g} is the gas pressure and P_{rad} is the radation pressure. Let $\beta = P_{\mathrm{g}}/P$.

(i) Calculate the specific heat c_{P} of this mixture in terms of β.

(ii) Define the adiabatic indices Γ_1, Γ_2 and Γ_3. Calculate Γ_1 and Γ_2 in terms of β. Write down an equation describing a relation between Γ_1, Γ_2 and Γ_3.

(iii) Calculate the ratio of specific heats $\gamma = c_{\mathrm{P}}/c_{\mathrm{V}}$ in terms of $(\Gamma_3 - 1)$ and derivatives of the pressure $\left(\dfrac{\partial \ln P}{\partial \ln T}\right)_{\varrho}$ and $\left(\dfrac{\partial \ln P}{\partial \ln \varrho}\right)_{\mathrm{T}}$.

(iv) What is the relation between γ and Γ_i, $i = 1, 2, 3$ for $\beta \to 0$ and $\beta \to 1$?

4

Consider now a spherically symmetric star of total mass M, luminosity L, which is in a radiative equilibrium. The star is made of perfect gas and is chemically homogeneous. The total pressure is provided by gas pressure $P_{\rm g}$ and radiation pressure $P_{\rm rad}$. The opacity of stellar material is given by

$$\kappa = \kappa_0 \, \frac{M_{\rm r}}{L_{\rm r}} \, \frac{L}{M}$$

where κ_0 is a constant, $M_{\rm r}$, $L_{\rm r}$ are the mass interior to radius r and the luminosity at radius r. You may assume that $P = 0, T = 0$ at the surface. Let $\beta = P_{\rm g}/P$.

(i) Find $T = T(\beta, \varrho)$ and $P = P(\beta, \varrho)$ for this star.

(ii) Prove that the star is a polytrope $P = K \, \varrho^{1+1/n}$ and that the polytropic constant K does not vary. Derive the appropriate Lane-Emden equation and state boundary conditions. Is it possible to solve this equation analytically in a closed form? Which stars might be approximated by this structure?

(iii) Prove that he luminosity cannot exceed the critical luminosity $L_{\rm crit} = \dfrac{4\,\pi\, c\, G\, M_{\rm r}}{\kappa}$.

(iv) Use homology to prove that the stellar mass obeys $M \propto \dfrac{(1 - \beta)^{1/2}}{\beta^2}$ (no credits will be given for other methods).

END OF PAPER