UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2022 1:30 pm to 4:30 pm

PAPER 315

EXTRASOLAR PLANETS: ATMOSPHERES AND INTERIORS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

a. Derive the condition for radiative equilibrium in a plane parallel atmosphere with an isotropic source function. Rewrite the condition under the assumption of local thermodynamic equilibrium and coherent isotropic scattering.

b. Consider the specific intensity in a plane parallel atmosphere given by the general form $I_\nu = A_\nu + B_\nu \cos^3\theta$, where ν is frequency and θ is the angle from the normal. A_ν and B_ν are isotropic quantities. Derive the relation between the mean intensity and the K-integral.

c. An isolated "free-floating" exoplanet was discovered to have a luminosity L and radius R. Starting with the radiative transfer equation, derive an expression for the temperature gradient in its atmosphere assuming radiative equilibrium and a grey opacity. Make and state any other assumptions required.

2

a. The following reactions are suggested to govern ozone chemistry in the atmosphere of an Earth-like exoplanet:

$$O_2 + h\nu \longrightarrow O + O \quad (1)$$
$$O + O_2 + M \longrightarrow O_3 + M \quad (2)$$
$$O_3 + h\nu \longrightarrow O + O_2 \quad (3)$$
$$O + O_3 \longrightarrow O_2 + O_2 \quad (4)$$

The photodissociation coefficients for reactions (1) and (3) are J_1 and J_3, respectively. The rate coefficients for reactions (2) and (4) are K_2 and K_4, respectively. Show how these reactions might lead to the presence of an ozone layer in the planet's atmosphere. You may assume that reaction (3) is the dominant mechanism for ozone destruction. Make and state any further assumptions required.

b. A planetary system has two giant exoplanets orbiting a sun-like star at orbital separations of 0.1 au and 5 au. The outer planet, of the same mass and radius as Jupiter, is undergoing hydrostatic thermal escape. Estimate a lower limit on the temperature at the exobase of the outer planet relative to that of an Earth-like planet at the same location. Make and state any assumptions needed.

The inner planet of mass $M_{\rm p}$ and radius $R_{\rm p}$ is undergoing hydrodynamic escape due to strong Extreme UV (EUV) radiation from its host star, with the incident EUV flux given by $F_{\rm EUV}$. The radius of the exobase relative to the planet centre is $R_{\rm exo}$ and a fraction η of the EUV flux is driving the hydrodynamic escape. Assuming energy-limited escape, derive an expression for the timescale over which a fraction x of the planet mass will be lost. Make and state any assumptions needed.

c. A transiting exoplanet of mass $M_{\rm p}$ and radius $R_{\rm p}$ orbiting a sun-like star at a separation a was observed at secondary eclipse over a wide wavelength range from 0.4-30 μm. The planet has an isothermal photosphere with a temperature $T_{\rm p}$ and a constant geometric albedo A_g. Derive an expression for the planet-star flux ratio that is applicable over the observed wavelength range. Make a sketch of the expression as a function of wavelength. Discuss the behaviour of the flux ratio at the two extremes of wavelength. Make and state any assumptions needed.

d. Derive a condition for thermochemical equilibrium in a layer of a planetary atmosphere at constant temperature and pressure.

3

a. For a spherically symmetric polytropic model of an exoplanet show that the total mass M and surface radius R are related as $R \propto M^{\beta}$. Determine β in terms of the polytropic index and discuss two values of β that are relevant in the planetary mass range. Make and state any assumptions required.

b. Derive the gravitational potential energy of a sphere of radius R and mass M of uniform density. Assuming a uniform density model, estimate the pressure at the centre of Neptune and Jupiter relative to that of the Earth.

c. A newly forming giant exoplanet of mass M and radius R orbits its host star at a separation of 90 au and radiates with a constant luminosity L. Assuming that the luminosity is primarily due to gravitational contraction, derive an expression for the age of the planet. Estimate the rate at which the radius of the planet is contracting. Make and state any assumptions required.

4

a. The luminosity of a sun-like star evolves with time t as $L = At^{-\beta}$, where A and β are positive constants. Using a single-layer atmospheric model derive an expression for the orbital separation a from the star at which a rocky exoplanet can maintain a surface temperature T_s. Make and state any assumptions required.

b. A transmission spectrum of a super-Earth observed between wavelengths of 1-2 μm revealed a low-amplitude H_2O feature equivalent to 2 scale heights assuming a solar abundance atmosphere for the corresponding equilibrium temperature. One explanation is the presence of a high-altitude cloud deck in the atmosphere. Estimate the pressure at the top of the cloud deck that could explain this observation, making any assumptions needed. What other explanations are possible? What other observations could help establish the right explanation.

c. The temperature gradient in the lower atmosphere of a giant exoplanet varies as $dT/dz = -K\rho/T^3$, where T is temperature, ρ is density, z is vertical distance and K is a constant. If this region of the atmosphere is stable against convection derive an expression for the pressure P_{rc} at the radiative-convective boundary. Estimate P_{rc} for an irradiated hot Jupiter. Make and state any assumptions required.

d. Consider the following chemical reaction in gas phase: A + B $\rightleftharpoons$ C + D. In chemical equilibrium, it is known that the forward reaction is favoured at low temperatures and the reverse reaction is favoured at high temperatures. However, atmospheric retrievals of a cool transiting exoplanet found an unexpectedly large amount of A. The retrievals also established the pressure and temperature at radius R_p of the planet to be P_0 and T_0, respectively. The timescale for this chemical reaction varies with height z in the atmosphere as $\tau_{chem} = \eta z$, where η is a constant. Estimate the quench pressure for A as a function of the eddy mixing coefficient K_{zz}. Make and state any assumptions required. List two examples of such processes in planetary atmospheres.

e. Discuss the effects of clouds/hazes on transmission and emission spectra of exoplanetary atmospheres. List three aspects of atmospheric dynamics that have been observed in giant exoplanets.

END OF PAPER