UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2022 9:00 am to 12:00 pm

PAPER 314

ASTROPHYSICAL FLUID DYNAMICS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{u} = 0, \tag{2}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\rho \nabla \Phi - \nabla p + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \tag{4}$$

$$\nabla^2 \Phi = 4\pi G \rho. \tag{5}$$

Conservation laws for momentum

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \hat{\Pi} = 0, \quad \hat{\Pi}_{ij} = \rho u_i u_j + \left(p + \frac{B^2}{8\pi} \right) \delta_{ij} - \frac{B_i B_j}{4\pi}, \tag{6}$$

and energy

$$\frac{\partial}{\partial t} \left[\rho \left(\frac{u^2}{2} + e \right) + \frac{B^2}{8\pi} \right] + \nabla \cdot \left[\rho \mathbf{u} \left(\frac{u^2}{2} + h \right) + c \frac{\mathbf{E} \times \mathbf{B}}{4\pi} \right] = 0, \tag{7}$$

where h is the enthalpy obeying $\mathrm{d}h = T\ \mathrm{d}s + \rho^{-1}\mathrm{d}p$; $h = c_s^2/(\gamma - 1)$ for a polytropic gas with adiabatic index γ, where c_s is the speed of sound.

You may assume that for any scalar function f

$$\nabla f = \frac{\partial f}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{e}_z \quad \textit{(cylindrical coordinates)} \tag{8}$$

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r \sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi \quad \textit{(spherical coordinates).} \tag{9}$$

You may assume that for any vector $\mathbf{C}$

$$(\nabla \times \mathbf{C}) \times \mathbf{C} = (\mathbf{C} \cdot \nabla)\mathbf{C} - \frac{1}{2}\nabla\left(|\mathbf{C}|^2\right), \tag{10}$$

and in cylindrical coordinates

$$\nabla \cdot \mathbf{C} = \frac{1}{R}\frac{\partial (R C_R)}{\partial R} + \frac{1}{R}\frac{\partial C_\phi}{\partial \phi} + \frac{\partial C_z}{\partial z}, \tag{11}$$

$$\nabla \times \mathbf{C} = \left(\frac{1}{R}\frac{\partial C_z}{\partial \phi} - \frac{\partial C_\phi}{\partial z} \right)\mathbf{e}_R + \left(\frac{\partial C_R}{\partial z} - \frac{\partial C_z}{\partial R} \right)\mathbf{e}_\phi + \frac{1}{R}\left[\frac{\partial (R C_\phi)}{\partial R} - \frac{\partial C_R}{\partial \phi} \right]\mathbf{e}_z. \tag{12}$$

For any two vectors $\mathbf{C}$ *and* $\mathbf{D}$

$$\nabla \times (\mathbf{C} \times \mathbf{D}) = \mathbf{C}(\nabla \cdot \mathbf{D}) + (\mathbf{D} \cdot \nabla)\mathbf{C} - \mathbf{D}(\nabla \cdot \mathbf{C}) - (\mathbf{C} \cdot \nabla)\mathbf{D}, \tag{13}$$

$$\nabla \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{D} \cdot (\nabla \times \mathbf{C}) - \mathbf{C} \cdot (\nabla \times \mathbf{D}). \tag{14}$$

You may refer to these formulae in your solutions, but, please, make sure to provide sufficient details when using them.

1

(a) Define what it means for a magnetostatic structure to be in *force-free* equilibrium. Show that the magnetic field structure in a force-free equilibrium is governed by the equation

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\alpha \boldsymbol{B},$$

where α is a function of spatial coordinates. Identify a constraint on the behaviour of $\alpha(\boldsymbol{r})$.

(b) Consider a magnetostatic force-free configuration in Cartesian $\boldsymbol{r} = (x, y, z)$ coordinates. Assume that $\boldsymbol{B}$ does not depend on y and z. Derive a closed form equation for the z-component of the magnetic field B_z for a general $\alpha(\boldsymbol{r}) = \alpha(x)$. Show that the constraint on the behaviour of $\alpha(\boldsymbol{r})$ is satisfied by this solution.

(c) Obtain a general solution of the equation derived in part (b) and determine all field components for a general $\alpha(x)$.

(d) Consider a particular form for $\alpha(x)$, namely

$$\alpha(x) = \frac{c}{4\pi}\frac{a^2 x}{(x^2 + d^2)^2},$$

where a and d are some constant length scales. Find the condition on d/a for the magnetic field components to not change sign anywhere in space, if it is also known that $B_z \to 0$ as $x \to \pm\infty$.

2

(a) Consider a steady, transonic flow of unmagnetized, ideal fluid. Let $\boldsymbol{j} = \rho\boldsymbol{u}$ be the fluid flux, where ρ and $\boldsymbol{u}$ are the fluid density and velocity, respectively. Neglecting gravity, find the expression for dj/du (where $j = |\boldsymbol{j}|$, $u = |\boldsymbol{u}|$) and use it to demonstrate the difference in the behaviour of j in the sub-sonic and super-sonic regimes.

[Hint: You may find it useful to project the equation of motion along the fluid streamline.]

(b) Show that j attains a maximum value at some point as u transitions from sub-sonic to super-sonic regimes. Let us denote the value of the sound speed of the flow at that point a *critical* speed $c_{\rm cr}$. How is the velocity of the flow at that point related to $c_{\rm cr}$?

(c) Assume that fluid is polytropic with adiabatic index γ. Establish a relation between the Bernoulli function of a streamline $C_{\rm B}$ and $c_{\rm cr}$, thereby demonstrating that $c_{\rm cr}$ is a unique characteristic of every streamline of the flow.

(d) Consider a planar shock in a uniform fluid flow, with no velocity parallel to the shock plane. Let u_1 and u_2 be the pre-shock and post-shock fluid velocities in the frame of the shock. Show that there is a unique relation between the product $u_1 u_2$ and the critical velocity of the flow $c_{\rm cr}$ and derive its explicit form.

3

The gravitational potential in the central region of a spherical galaxy is dominated by a static dark matter configuration with the density distribution

$$\rho_{\mathrm{dm}}(r) = \frac{\Psi_0}{r^{5/2}},$$

where $\Psi_0 > 0$ is a constant and r is the distance from the center of the galaxy. Dark matter particles do not interact with normal matter apart from their gravitational coupling.

(a) Derive the gravitational potential $\Phi_{\mathrm{dm}}(r)$ due to the dark matter configuration alone.

(b) The central part of the galaxy also contains unmagnetized gas, which is uniform and has density ρ_0 and sound speed c_0 far from the galactic centre. The gas is polytropic, with adiabatic index $\gamma > 1$. A small black hole of mass M residing at $r = 0$ accretes this gas in a spherically-symmetric fashion. Neglecting the gravity of the black hole, demonstrate that the accretion flow admits a sonic point as long as γ satisfies a constraint that is to be determined explicitly. You may assume that the accretion flow is time independent.

(c) Assuming that accretion is transonic, determine the radius of the sonic point. Find the mass accretion rate onto the black hole $\dot{M}$ and comment on the difference of $\dot{M}$ dependence on c_0 compared to the case of classical Bondi accretion.

(d) Find the condition on the black hole mass for the subsonic part of the accretion flow to be only weakly affected by the gravitational effect of the black hole, justifying the assumption made in part (b).

4

The cross-helicity of a magnetized flow H_{c} is defined such that its volumetric density is $h_{\mathrm{c}} = \boldsymbol{u} \cdot \boldsymbol{B}$, where $\boldsymbol{u}$ is the velocity and $\boldsymbol{B}$ is the magnetic field.

(a) Formulate a conservation law for the cross-helicity, i.e. find an explicit form for the cross-helicity density flux $\boldsymbol{F}_{\mathrm{c}}$ and the source term S_{c} such that

$$\frac{\partial h_{\mathrm{c}}}{\partial t} + \nabla \cdot \boldsymbol{F}_{\mathrm{c}} = S_{\mathrm{c}},$$

and S_{c} vanishes for the homentropic flow (i.e. the one with spatially uniform entropy).

(b) Consider a time-dependent, magnetized, homentropic flow, in which the velocity $\boldsymbol{u}$ is everywhere parallel to the magnetic field $\boldsymbol{B}$. Show that in such a flow the helicity

$$H_{\mathrm{c}} = \int_V h_{\mathrm{c}} \, \mathrm{d}V$$

is conserved in any volume V as long as the the Bernoulli function, as defined for a steady, unmagnetized flow, does not vary along the magnetic field lines.

END OF PAPER