UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2022 1:30 pm to 3:30 pm

PAPER 313

SOLITONS, INSTANTONS, AND GEOMETRY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Use the Derrick scaling argument to show that if ϕ satisfies the field equation arising from extremising the functional

$$E[\phi] = \int_{-\infty}^{\infty} \Big(\frac{1}{2}\Big(\frac{d\phi}{dx}\Big)^2 + U(\phi)\Big) dx$$

and is such that $E[\phi]$ is finite then

$$\int_{-\infty}^{\infty} \frac{1}{2}\Big(\frac{d\phi}{dx}\Big)^2 dx = \int_{-\infty}^{\infty} U(\phi) dx.$$

Find the potential $U(\phi)$ corresponding to the static field equation

$$\frac{d^2\phi}{dx^2} = 3\phi^5 - 4\phi^3 + \phi,$$

and such that the minimum of U is zero. How many elements are there in the set $U^{-1}(0)$? How many types of kink solution exist for this potential?

Find the Bogomolny equation satisfied by one of these kink solutions, and construct the kink explicitly.

Using the Bogomolny argument or otherwise find the mass of this kink.

2

The potential energy functional of the Abelian Higgs model is

$$E[\phi, A] = \int_{\mathbb{R}^2} \Big(\frac{1}{2}B^2 + \frac{1}{2}(D_1\bar{\phi}D_1\phi + D_2\bar{\phi}D_2\phi) + \frac{1}{8}(1 - |\phi|^2)^2\Big) dx^1 dx^2, \qquad (1)$$

where $\phi : \mathbb{R}^2 \to \mathbb{C}$, the one–form A is the Abelian potential with the covariant derivative $D = d - iA$, and B is the magnetic field such that $dA = B dx^1 \wedge dx^2$.

Define a vortex number, and show how it is related to the magnetic flux $\int_{\mathbb{R}^2} B dx^1 \wedge dx^2$.

Consider an ansatz

$$A = f(r) d\theta, \quad \phi = h(r) e^{ik\theta}, \quad \text{where} \quad x^1 + ix^2 = re^{i\theta}, \quad k \in \mathbb{N}$$

and define $\mathcal{E}(f, h)$ to be the functional resulting from substituting this into (1). Show that $\mathcal{E} \geqslant k\pi$, with the equality iff f, h satisfy a coupled system of ODEs

$$h' = \frac{1}{r}(k - f)h, \quad f' = \frac{r}{2}(1 - h^2).$$

What are the boundary conditions for f, h?

3

Define a Hodge operator $*$ on $(\mathbb{R}^n, \mathrm{vol}, \eta)$, where η is a pseudo–Riemannian metric, and vol is the volume form. Show that if $\sigma, \mu \in \Lambda^2(\mathbb{R}^4)$ are such that $*\sigma = \sigma, *\mu = -\mu$ then $\sigma \wedge \mu = 0$.

Consider $n = 4$ together with

$$\eta = dw d\bar{w} + dz d\bar{z}, \quad \mathrm{vol} = dw \wedge dz \wedge d\bar{w} \wedge d\bar{z} \quad \text{where} \quad (w, z) \in \mathbb{C}^2$$

and show that the real two–forms $\omega_k, k = 1, 2, 3$ defined by

$$\omega_1 + i\omega_2 = dw \wedge dz, \quad \omega_3 = i(dw \wedge d\bar{w} + dz \wedge d\bar{z})$$

are self–dual.

Construct a Lax pair for the anti–self–dual Yang–Mills (ASDYM) equations. Deduce that if these equations hold in the $(w, z, \bar{w}, \bar{z})$ coordinates, then there exists a gauge such that

$$A_w = A_z = 0,$$

where $A = A_w dw + A_z dz + A_{\bar{w}} d\bar{w} + A_{\bar{z}} d\bar{z}$ is the Yang–Mills potential.

END OF PAPER