UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 13 June, 2022 9:00 am to 12:00 pm

PAPER 312

FIELD THEORY IN COSMOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

When expanded to quadratic order in scalar field fluctuations φ, a particular model of inflation is found to give the action,

$$S_2[\varphi] = \int dt\, d^3\mathbf{x}\, \frac{a^3}{2} \left(\dot{\varphi}^2 - c^2 (\partial_i \varphi)^2 - 2H^2 \varphi^2 \right) \ ,$$

where the parameters c and $H = \dot{a}/a$ may be treated as constants.

(a) Show that the free equation of motion for φ in momentum space is,

$$\left(\partial_\tau^2 + c^2 k^2 \right) a\, \varphi_{\mathbf{k}} = 0 \ ,$$

when written in terms of conformal time $d\tau = dt/a$.

(b) Show that the mode functions in this free theory take the form,

$$f_k(\tau) = \mathcal{N}\, \tau \exp\left(-ick\tau \right) \ ,$$

and determine the overall normalisation, $\mathcal{N}$.

(c) Find the power spectrum, $\langle \varphi_{\mathbf{k}} \varphi_{\mathbf{k}'} \rangle$, in this free theory at any finite time τ.

(d) When further expanded to quartic order in φ, this model of inflation is found to contain the interaction,

$$S_4[\varphi] = \int dt\, d^3x \frac{a^3}{4!}\, \lambda \varphi^4 \ .$$

Find the trispectrum $\langle \varphi_{\mathbf{k}_1} \varphi_{\mathbf{k}_2} \varphi_{\mathbf{k}_3} \varphi_{\mathbf{k}_4} \rangle$ at time τ to leading order in λ.

(e) Under which of the ten de Sitter isometries do you expect these correlators to be invariant?

2

A massless vector field A_μ on an FLRW spacetime background is described by the quadratic action,

$$S_2[A_\mu] = \int dt\, d^3\mathbf{x}\, a^3 \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right] ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is invariant under the gauge symmetry,

$$A_\mu \to A_\mu + \partial_\mu \epsilon \ .$$

(a) Separate A_μ into two 3d scalars and a transverse 3d vector A_i^V, and find how each of these 3d objects transforms under the gauge symmetry. Identify a combination of your 3d scalars which is gauge-invariant. You may assume that $\partial_i \partial_i \epsilon \neq 0$.

(b) By performing the same 3d decomposition in $S_2[A_\mu]$, find the constraint equation which fixes this gauge-invariant scalar.

(c) Consider the gauge transformation with parameter,

$$\epsilon(t, \mathbf{x}) = b_i(t) \mathbf{x}^i \ . \tag{1}$$

Find the new field configuration generated by applying this transformation to the trivial solution, $A_\mu = 0$. What condition on $b_i(t)$ would allow this to be deformed into a physical perturbation?

(d) Assuming now that b_i is a constant, write down the generator Q of the symmetry transformation (1) in terms of Π^i, the momentum conjugate to A_i^V. Treating Π^i and A_i^V as free fields with mode functions $f_k(t)$, show that when acting on the vacuum,

$$Q|0\rangle = \lim_{\mathbf{q}\to 0} \left[\frac{a^3 \partial_t f_q^*}{f_q^*} \, b^i A_i^V(\mathbf{q}) |0\rangle \right] .$$

(e) Using the Ward identity associated with this gauge transformation, find a relation between the correlators $\langle \mathcal{O} A_i^V \rangle$ and $\langle A_i^V \mathcal{O} \rangle$ in the limit where the A_i^V momentum vanishes, where $\mathcal{O}$ is any neutral operator (invariant under the gauge symmetry). How would your result change if $\mathcal{O}$ were charged under the gauge symmetry?

3 (a) In a simple model of nonlinear biasing, the galaxy distribution $\delta_{\rm g}$ is given as a term proportional to the linear density perturbation δ (a Gaussian Random Field) plus the squared quantity δ^2 as

$$\delta_{\rm g}(\mathbf{x}) = b_1\delta(\mathbf{x}) + \tfrac{1}{2}b_2\big(\delta^2(\mathbf{x}) - \langle\delta^2(\mathbf{x})\rangle\big)\,,$$

where b_1, b_2 are bias parameters. By first finding an expression for the second-order solution $\delta_{\rm g}(\mathbf{k})$ in Fourier space, show the leading contribution to the bispectrum is

$$\langle\delta(\mathbf{k}_1)\,\delta(\mathbf{k}_2)\,\delta(\mathbf{k}_3)\rangle = b_1^2 b_2(2\pi)^3\delta^{(\rm D)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\big(P(k_1)P(k_2) + \text{cyclic perms}\big)\,,$$

where the power spectrum is $\langle\delta(\mathbf{k}_1)\,\delta(\mathbf{k}_2)\rangle = (2\pi)^3 P(k_1)\,\delta(\mathbf{k}_1+\mathbf{k}_2)$ with $k = |\mathbf{k}|$.

(b) The collisional Boltzmann equation for the CMB can be expressed in the form

$$\frac{d\Theta}{d\tau} = -\frac{d\Psi}{d\tau} + \Phi' + \Psi' + \Gamma\left(\Theta_0 - \Theta - \hat{\mathbf{n}}\cdot\mathbf{v}_{\rm e}\right)\,, \tag{$\dagger$}$$

where Θ is the temperature perturbation, Θ_0 is its monopole (without directional dependence), Φ and Ψ are the gravitational potentials in the Newtonian gauge, $\mathbf{v}_{\rm e}$ is the electron velocity and $\hat{\mathbf{n}}$ is the line of sight. Here, $\Gamma(\tau)$ is the electron collision rate which yields the optical depth $\mathcal{T}(\tau) \equiv \int_\tau^{\tau_0} d\tau'\,\Gamma(\tau')$ and the visibility function $g(\tau) \equiv -\mathcal{T}'e^{-\mathcal{T}}$.

(i) Use an integrating factor to find an integral solution of ($\dagger$), then assume instantaneous recombination, $g(\tau) = \delta^{(\rm D)}(\tau - \tau_{\rm dec})$, to derive the Sachs-Wolfe formula (at $\mathbf{x}_0 = 0$):

$$\Theta(\tau_0, \mathbf{x}_0 = 0, \hat{\mathbf{n}}) = \Theta_{0,\rm dec} + \Psi_{\rm dec} - \hat{\mathbf{n}}\cdot\mathbf{v}_{\rm e} + \int_{\tau_{\rm dec}}^{\tau_0} d\tau\,(\Phi' + \Psi')\,.$$

Briefly describe each physical contribution to the CMB temperature anisotropy.

(ii) The Fourier transform of the Boltzmann equation ($\dagger$) can also be written as

$$\Theta' + ik\mu\Theta = \Phi' - ik\mu\Psi + \Gamma(\Theta_0 - \Theta + i\mu v_e)\,, \tag{$*$}$$

where $\mu = \hat{\mathbf{n}}\cdot\hat{\mathbf{k}}$. Axisymmetry means that the directional dependence of the Boltzmann equation can be expanded in Legendre polynomials $P_\ell(\mu)$ as

$$\Theta(\tau, \mathbf{k}, \hat{\mathbf{n}}) = \sum_{\ell=0}^{\infty}(-i)^\ell(2\ell+1)\,\Theta_\ell(\tau,\mathbf{k})\,P_\ell(\mu)\,,$$

where $\Theta_\ell(\tau,\mathbf{k}) = \frac{1}{(-i)^\ell}\int\frac{d\mu}{2}P_\ell(\mu)\,\Theta(\tau,\mathbf{k},\hat{\mathbf{n}})$. Derive the Boltzmann hierarchy from ($*$) to find the expressions:

$$\begin{aligned}
&\Theta_0' + k\Theta_1 = \Phi'\,,\\
&3\Theta_1' + k\,(\Theta_2 - \Theta_0) = k\Psi - \Gamma(3\Theta_1 + v_{\rm e})\,,\\
&\Theta_\ell' + \frac{k}{2\ell+1}\left((\ell+1)\Theta_{\ell+1} - \ell\Theta_{\ell-1}\right) = -\Gamma\Theta_\ell\,, \quad (\text{for } \ell \geqslant 2)\,.
\end{aligned}$$

[*You may use the recursion relation* $(2\ell+1)\mu\,P_\ell(\mu) = (\ell+1)P_{\ell+1}(\mu) + \ell\,P_{\ell-1}(\mu)$ *and the orthogonality condition* $\int_{-1}^{1} d\mu\,P_\ell(\mu)\,P_{\ell'}(\mu) = 2\,\delta_{\ell\ell'}/(2\ell+1)$.]

4 (i) Assume that the matter density perturbation $\delta(\mathbf{x},\tau)$ and velocity $\mathbf{v}(\mathbf{x},\tau)$ satisfy

$$\delta' + \nabla\cdot[(1+\delta)\,\mathbf{v}] = 0\,, \qquad\qquad v_i' + \mathcal{H}v_i + \mathbf{v}\cdot\nabla v_i = -\nabla_i\phi - \rho^{-1}\,\nabla_j\,(\rho\,\sigma_{ij})\,, \qquad (*)$$

where ρ is the matter density, σ_{ij} is the anisotropic stress tensor, and the gravitational potential ϕ obeys $\nabla^2\phi = \frac{3}{2}\mathcal{H}\delta$, with $\mathcal{H} = a'/a$ in an Einstein-de Sitter universe with scale factor a. You are given the following second-order perturbative solution in Fourier space

$$\tilde{\delta}^{(2)}(\mathbf{k}) = \int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\int \frac{d^3\mathbf{k}_2}{(2\pi)^3}\, F_2(\mathbf{k}_1,\mathbf{k}_2)\,\tilde{\delta}^{(1)}(\mathbf{k}_1)\,\tilde{\delta}^{(1)}(\mathbf{k}_2)\,(2\pi)^3\delta^{(\mathrm{D})}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\,, \qquad (\dagger)$$

where a power series ansatz for δ and $\theta = \nabla\cdot\mathbf{v}$ defines terms at each order

$$\delta(\mathbf{k},\tau) = a(\tau)\,\tilde{\delta}^{(1)}(\mathbf{k}) + a^2(\tau)\,\tilde{\delta}^{(2)}(\mathbf{k}) + \dots\,, \quad \theta(\mathbf{k},\tau) = -\mathcal{H}(\tau)\left[a\,\tilde{\theta}^{(1)}(\mathbf{k}) + a^2\,\tilde{\theta}^{(2)}(\mathbf{k}) + \dots\right],$$

and the two-point coupling kernel $F_2(\mathbf{k}_1,\mathbf{k}_2)$ is given by

$$F_2(\mathbf{k}_1,\mathbf{k}_2) = \frac{1}{14}\,\frac{1}{k_1^2\,k_2^2}\left[10\,k_1^2\,k_2^2 + 7\,(\mathbf{k}_1\cdot\mathbf{k}_2)\left(k_1^2+k_2^2\right) + 4\,(\mathbf{k}_1\cdot\mathbf{k}_2)^2\right]\,.$$

Briefly state the underlying assumptions that have been used to obtain the second-order solution ($\dagger$). [*You are not required to derive* ($\dagger$).]

(ii) Using Feynmann diagrams (or otherwise) write down integral expressions for the one-loop power spectrum in terms of the linear solutions and two-point $F_2(\mathbf{k}_1,\mathbf{k}_2)$ and three-point $F_3(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ coupling kernels (assumed to be given), that is, specify the following:

$$P_{\text{1-loop}}(k) = a^2 P(k) + a^4\left[P_{22}(k) + 2P_{13}(k)\right]\,,$$

where the linear power spectrum is given by $\langle\tilde{\delta}^{(1)}(\mathbf{k}_1)\,\tilde{\delta}^{(1)}(\mathbf{k}_2)\rangle = (2\pi)^3\delta^{(\mathrm{D})}(\mathbf{k}_1+\mathbf{k}_2)\,P(k)$.

(iii) Show that the $P_{22}(k)$ power spectrum can be written with $\mu = \hat{\mathbf{k}}\cdot\hat{\mathbf{q}}$ in the form

$$P_{22}(k) = \int \frac{dq}{4\pi^2}\int_{-1}^{1} d\mu\,\frac{k^4\left(7k\mu + q(3-10\mu^2)\right)^2}{98\left(k^2 - 2kq\mu + q^2\right)^2}P(q)P\left(\sqrt{k^2 - 2\mu kq + q^2}\right)\,.$$

Hence, show the infrared limit (IR: $q\to 0$ or $|\mathbf{k}-\mathbf{q}|\to 0$) and ultraviolet limit (UV: $q \gg k$) of the $P_{22}(k)$ integrand contributions can be represented as:

$$P_{22,\mathrm{IR}}(k) \longrightarrow \frac{1}{3}k^2 P(k)\int\frac{dq}{2\pi^2}P(q)\,, \qquad\qquad P_{22,\mathrm{UV}}(k) \longrightarrow \frac{9}{98}k^4\int\frac{dq}{2\pi^2}\frac{P^2(q)}{q^2}\,.$$

(iv) You are given that the IR and UV contributions for the $P_{13}(k)$ power spectrum are

$$P_{13,\mathrm{IR}}(k) \longrightarrow\ -\frac{1}{6}k^2 P(k)\int\frac{dq}{2\pi^2}P(q)\,, \qquad\qquad P_{13,\mathrm{UV}}(k) \longrightarrow -\frac{61}{630}k^2 P(k)\int\frac{dq}{2\pi^2}P(q)\,.$$

Using the asymptotic properties of P_{13} and P_{22}, discuss the leading-order counterterm required to remove the cut-off dependence of the one-loop power spectrum in the Effective Field Theory of Large-Scale Structure. Provide some physical motivation for introducing this counterterm in the context of the dynamical equations ($*$).

END OF PAPER