UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 8 June, 2022 9:00 am to 12:00 pm

PAPER 311

BLACK HOLES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A spacetime containing a static, spherically symmetric, star has line element

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \dfrac{2\,m(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,,$$

where $\theta \in [0, \pi]$ and $\phi \sim \phi + 2\pi$ are the usual angles on the unit round two sphere.

The matter inside the star is described by a perfect fluid with energy momentum tensor $T_{ab} = (\rho + p)u_a u_b + p\, g_{ab}$ and equation of state $p = p(\rho)$ with $\rho, p \geqslant 0$, $dp/d\rho > 0$. The Einstein equation reduces to the Tolman-Oppenheimer-Volkov (TOV) equations:

$$\begin{aligned}\frac{dm}{dr} &= 4\pi r^2 \rho\,,\\ \frac{d\Phi}{dr} &= \frac{m + 4\pi r^3 p}{r(r-2m)}\\ \frac{dp}{dr} &= -(p+\rho)\frac{m + 4\pi r^3 p}{r(r-2m)}\,.\end{aligned}$$

(a) (i) Let R denote the radius of the star, so p, ρ vanish for $r > R$. Show that the metric outside the star is the Schwarzschild metric. [4]

(ii) Explain why smooth solutions of the TOV equations form a 1-parameter family, labelled uniquely by $\rho_c \equiv \rho(0)$. [6]

(iii) Assume that the equation of state is known for $\rho \leqslant \rho_0$ but not for $\rho > \rho_0$. Explain why there is a maximum possible mass for the star that is independent of the equation of state for $\rho > \rho_0$. [6]

You may assume that a solution of the TOV equations satisfies

$$\frac{m(r)}{r} < \frac{2}{9}\left[1 - 6\pi r^2 p + (1 + 6\pi r^2 p)^{1/2}\right]\,.$$

(b) Now assume that the star has constant density $\rho = \rho_0$ for $0 \leqslant r \leqslant R$.

(i) Show that for $0 \leqslant r \leqslant R$

$$p(r) = \rho_0 \frac{\sqrt{1 - \dfrac{2M}{R}} - \sqrt{1 - \dfrac{2M\,r^2}{R^3}}}{\sqrt{1 - \dfrac{2Mr^2}{R^3}} - 3\sqrt{1 - \dfrac{2M}{R}}}\,.$$

[8]

(ii) The matter at the centre of the star obeys a linear barotropic bound $p(0) \leqslant \omega\,\rho(0)$, with $\omega > 0$. Derive an upper bound on M/R for a constant density star satisfying this condition and show that such stars can get arbitrarily close to saturating Buchdahl's inequality. [6]

2 The metric and gauge potential of a spherically symmetric isolated charged gravitating object in $d \geqslant 4$ spacetime dimensions is, in Schwarzschild-like coordinates,

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2 \mathrm{d}\Omega_{d-2}^2\,, \quad A = -\frac{Q}{r^{d-3}}\,\mathrm{d}t\,, \tag{$\star$}$$

where $\mathrm{d}\Omega_{d-2}^2$ is the line element on a unit radius $(d-2)$ round sphere, defined iteratively as

$$\begin{aligned} \mathrm{d}\Omega_1^2 &= \mathrm{d}\phi^2\,, \\ \mathrm{d}\Omega_{n+1}^2 &= \mathrm{d}\theta_n^2 + \sin^2\theta_n \mathrm{d}\Omega_n^2\,, \end{aligned}$$

with $\theta_i \in [0,\pi]$ for $i = 1, \ldots, d-3$ and ϕ a periodic coordinate with period 2π. Furthermore,

$$f(r) = \left(1 - \frac{r_+^{d-3}}{r^{d-3}}\right)\left(1 - \frac{r_-^{d-3}}{r^{d-3}}\right)\,,$$

where $0 \leqslant r_- \leqslant r_+$ and $Q = r_+^{\frac{d-3}{2}} r_-^{\frac{d-3}{2}}$.

(a) Show that

$$K = \frac{\partial}{\partial t}$$

is a Killing vector field. [2]

(b) Construct the analogue of ingoing Eddington-Finkelstein coordinates and determine the form of the metric in these coordinates. Explain briefly why this metric can be analytically extended across the surface $r = r_+$. [4]

(c) Show that $r = r_+$ is a null hypersurface and is a Killing horizon of the Killing vector field K and determine its associated surface gravity. [6]

(d) Show that $r = 0$ is not a coordinate singularity and sketch the Penrose diagrams, carefully distinguishing between the cases $r_- = 0$, $0 < r_- < r_+$ and $r_- = r_+$. [6]

(e) The equations of motion for a particle with mass m, electric charge q and $4-$velocity U are

$$U^a \nabla_a U^b = \frac{q}{m} F^b{}_c U^c\,.$$

(i) Show that

$$E = -m\,K \cdot U + q\,\Phi$$

is conserved along the particle's worldline with $i_K F = \mathrm{d}\Phi$. [4]

(ii) Choose $\lim_{r+\infty} \Phi = 0$. Show that if q and Q have opposite signs, E can become negative for a future-directed particle held at fixed $(r, \theta_1, \ldots, \theta_2, \phi)$, with r sufficiently close to r_+. Explain why this fact can be used to extract energy from the black hole. [4]

(iii) Obtain an upper limit for the amount of energy that can be extracted from this process and show that this upper limit agrees with that obtained from the area theorem. *[You may assume that the mass of the spacetime is given by $M = \frac{r_+^{d-3} + r_-^{d-3}}{2}$ and that a particle of charge q and mass m crossing the black hole event horizon will charge the black hole charge by an amount $\delta Q = q$.]* [4]

3 Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime with a Cauchy surface S. Assume that the Einstein equation and the *strong energy condition* are satisfied. Introduce a function t defined near S, so that $t = 0$ on S and let $t_a = \nabla_a t$ have $t^a t_a = -1$ and t^a be future-directed.

(a) Show that t^a is a geodesic vector field emanating normally from S. [2]

(b) Let $c = -\nabla_a t^a$ be the *convergence* of t^a. Show that

$$t^a \nabla_a c = R_{ab} t^a t^b + (\nabla_a t_b)(\nabla^a t^b)\,,$$

where R_{ab} are the components of the Ricci tensor. [4]

(c) Use the results in part (b) to show that

$$t^a \nabla_a c \geqslant \frac{1}{3} c^2\,.$$

[Hint: consider expanding $S^{ab} S_{ab}$, where $S_{ab} = \nabla_a t_b - \frac{1}{3}(g_{ab} + t_a t_b)\nabla_c t^c$.] [10]

(d) Show that if $c = c_0 > 0$ at some initial point on the geodesic, then c must become infinite at least by a further proper distance $3/c_0$ along the geodesic. [4]

(e) Let c be bounded below on S by some positive constant c_0. Show that $(\mathcal{M}, g)$ must be timelike geodesically incomplete.

[You may assume that if p lies in the domain of dependence of S there will exist a timelike curve from p to S of maximal length and that if c diverges, nearby timelike geodesics intersect.] [10]

4 Write an essay on the relation of black holes to thermodynamics.

You should start with a statement of the laws of black hole mechanics and explain why they are analogous to the laws of thermodynamics. You should explain how the connection between surface gravity and temperature, together with the first law of black hole mechanics, leads to the Bekenstein-Hawking formula for the entropy of a black hole.

Next, you should consider a quantum scalar field Φ satisfying the wave equation $\nabla_a \nabla^a \Phi = 0$ in a globally hyperbolic non-stationary spacetime that is asymptotic to Minkowski spacetime in the far past and far future (*i.e.* a sandwich spacetime), and explain how the vacuum state can evolve to a non-vacuum state. This discussion should carefully describe the quantization of the scalar field, why the notion of particle is ambiguous and discuss particle production in a sandwich spacetime. You should then explain briefly how your results apply to late-time Hawking radiation from a Schwarzschild black hole formed from gravitational collapse.

You should conclude with a brief discussion of some of the implications of Hawking radiation for black holes. [30]

END OF PAPER