UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2022 1:30 pm to 4:30 pm

## PAPER 310

## COSMOLOGY

**Before you begin please read these instructions carefully**

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

**STATIONERY REQUIREMENTS**
Cover sheet
Treasury tag
Script paper
Rough paper

**SPECIAL REQUIREMENTS**
None

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** **(a)** From the continuity equation, determine the scaling of the energy density $\rho$ with scale factor $a$ for a component with constant equation of state parameter $w$. Hence show that the Hubble parameter can be written as $H(z) = H_0 E(z)$, with

$$E(z) = \left[ \sum_i \Omega_{i,0} (1+z)^{3(1+w_i)} \right]^{1/2} , \tag{1}$$

where you should define $\Omega_{i,0}$ and where the sum is over components $i$ with constant equation of state parameters $w_i$. Explain how the expansion of the universe, parametrized by $a(t)$, can be determined given $\Omega_{i,0}$ and $w_i$ for all components.

**(b)** Now consider a universe containing only cold dark matter (m) and a non-standard dark energy component (DE); the non-standard dark energy component has an equation of state parameter that is not constant, but instead depends on redshift in the following way:

$$w(z) = w_0 + \frac{w_a z}{1+z}, \tag{2}$$

where $w_0$ and $w_a$ are constant parameters. Carefully show that in such a universe the Hubble parameter is given by

$$H(z) = H_0 \left[ \Omega_{m,0}(1+z)^3 + X(\Omega_{DE,0}, z, w_0, w_a) \right]^{1/2} , \tag{3}$$

where $X(\Omega_{DE,0}, z, w_0, w_a)$ is a function you should specify.

**(c)** A bright source in a distant galaxy emits photons at a time $t_s$ that are received on earth at time $t_0$. The source's redshift $z_1$ is initially measured by astronomers at a time $t_0$. The redshift of the same source is measured a second time, giving $z_2$, after waiting an interval $\Delta t_0$= 10 years (which corresponds to an interval of $\Delta t_s$ in the rest frame of the source). Argue first that the difference $\Delta z \equiv z_2 - z_1$ (referred to as the redshift drift) between the redshifts of the source at times $t_0 + \Delta t_0$ and $t_0$ is given by

$$\Delta z = \frac{a(t_0 + \Delta t_0)}{a(t_s + \Delta t_s)} - \frac{a(t_0)}{a(t_s)}. \tag{4}$$

After relating $\Delta t_s$ to $\Delta t_0$, show that the redshift drift $\Delta z$ is given by

$$\Delta z = f(z_1, \Delta t_0, E(z_1), H_0), \tag{5}$$

where you should determine the function $f(z_1, \Delta t_0, E(z_1), H_0)$. You may perform all calculations to linear order in $H\Delta t \sim \Delta t/t \ll 1$.
*[Hint: to relate $\Delta t_s$ to $\Delta t_0$, you may wish to argue that the conformal time elapsed must be the same at emission and observation]*

**(d)** Could such measurements of source redshift drifts be used to simultaneously determine $H_0$, $\Omega_{m,0}$, and $\Omega_{DE,0}, w_0, w_a$ (the parameters of the non-standard dark energy model from part (b))? Briefly justify your answer.

**2** In this question you will discuss recombination and the decoupling of CMB photons in the early universe.

**(a)** At high temperatures ($T > 1\text{eV}$), electrons, protons and neutral Hydrogen are in equilibrium due to reactions such as $e^- + p \leftrightarrow H + \gamma$. Show that the equilibrium number density $n_H$ of neutral Hydrogen is given by the equation

$$\left(\frac{n_H}{n_e^2}\right)_{eq} = \left(\frac{2\pi}{m_e T}\right)^{3/2} e^{B_H/T}, \tag{1}$$

where $n_e$ is the free electron number density, $B_H = m_e + m_p - m_H$ is the binding energy of Hydrogen, and $m_e, m_p, m_H$ are the electron, proton, and neutral Hydrogen masses. You may assume charge neutrality of the universe.

*[Hint: you may assume that the equilibrium number density for non-relativisic particles is* $n_i^{\text{eq}} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T}\right)$*]*

**(b)** Hence derive the Saha equation, which describes the recombination process in equilibrium:

$$\left(\frac{1 - X_e}{X_e^2}\right)_{eq} = \frac{2\zeta(3)}{\pi^2}\eta \left(\frac{2\pi T}{m_e}\right)^{3/2} e^{B_H/T}, \tag{2}$$

where $X_e \equiv \frac{n_e}{n_b}$ is the free electron fraction, $n_b$ is the number density of baryons, and $\eta \equiv \frac{n_b}{n_\gamma} \sim 10^{-9}$ is the baryon-photon ratio. You may neglect all nuclei other than protons (so that $n_b \approx n_p + n_H$, where $n_p$ is the proton number density). You may also assume that the photon number density is given by $n_\gamma(T) = \frac{2\zeta(3)}{\pi^2} T^3$.

Define the recombination temperature $T_{rec}$ as when $X_e$ falls below $X_e = 0.1$. Evaluating the Saha equation gives $T_{rec} \approx 0.3eV \approx 3600K$. Why is this temperature much lower than the naive recombination temperature $T \approx B_H \approx 13.6\text{eV}$?

**(c)** Photons interact with the primordial plasma primarily via their interactions with electrons. You may assume that the interaction rate is $\Gamma_\gamma \approx n_e \sigma_T$, where $\sigma_T$ is the Thomson cross-section. State the criterion for decoupling of photons from the primordial plasma to occur and show that this takes place at an approximate temperature $T_{dec}$ specified by:

$$X_e(T_{dec}) T_{dec}^{3/2} \sim \frac{\pi^2}{2\zeta(3)} \frac{H_0 \sqrt{\Omega_{m,0}}}{\eta \sigma_T T_0^{3/2}} \tag{3}$$

where $T_0$ is the CMB temperature today. Evaluating this equation for the decoupling temperature numerically, the result is $T_{dec} \approx 0.27\text{eV}$. $T_{dec}$ is found to be only very weakly dependent on parameters such as $H_0$ or $\Omega_m$ on the RHS of Equation (3) – motivate this briefly based on the form of the equations you have derived (you may assume that $X_e(T_{dec}) \ll 1$). Due to this result, in the remainder of the question you may approximate $T_{dec}$ as $T_{dec} \approx T_{rec}$.

**(d)** Imagine that our current measurements of the CMB temperature $T_0$ were incorrect and that the current CMB temperature was actually $T_0 = 1K$ instead of $T_0 = 2.73\text{K}$. Assuming $T_{dec} = T_{rec}$ and that $\eta$ is fixed, determine the ratio of decoupling temperatures $T_{dec,1K}/T_{dec,2.73K}$ and redshifts $z_{dec,1K}/z_{dec,2.73K}$ between cosmologies where $T_0 = 1K$ and where $T_0 = 2.73\text{K}$. What parameters could be varied to keep the comoving distance to CMB decoupling fixed, despite a change in $T_0$?

**3** In this question you will discuss the CMB lensing signal as a direct gravitational probe of the matter power spectrum. You may assume throughout that all relevant scales are sub-horizon. You may also assume that linear perturbation theory is sufficiently accurate for all calculations.

**(a)** Starting from the evolution equation

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G(\bar{\rho}_m\delta_m + \bar{\rho}_r\delta_r) = 0, \qquad (1)$$

derive how $\delta_m$ evolves with the scale factor $a$ during matter domination and radiation domination (you may assume that, since $\delta_r$ oscillates rapidly during radiation domination, it can be neglected for the purposes of determining the growth of $\delta_m$).

**(b)** Sketch the shape of the matter power spectrum $P(k)$ and explain briefly why the slow sub-horizon growth of $\delta_m$ during radiation domination is important for determining this shape.

**(c)** Due to gravitational lensing, the CMB anisotropies are remapped by an observable angle $\nabla\psi(\hat{\mathbf{n}})$, where $\psi$ is the CMB lensing potential and $\hat{\mathbf{n}}$ is a direction on the sky. This CMB lensing potential is related in the following way to the Newtonian potential perturbation $\phi$:

$$\psi(\hat{\mathbf{n}}) = -2\int_0^{\chi_e} \frac{\chi_e - \chi}{\chi\chi_e}\phi(\hat{\mathbf{n}}\chi, \tau(\chi))d\chi. \qquad (2)$$

Here $\phi(\mathbf{x}, \tau)$ is the Newtonian-gauge potential perturbation evaluated at position $\mathbf{x}$ and conformal time $\tau$; $\chi$ is the comoving distance (which corresponds to the conformal time on the photons path $\tau(\chi) = \tau_0 - \chi$), and $\chi_e$ is the distance from Earth to the point where the CMB photon was emitted. You may assume that the Newtonian potential perturbation $\phi$ and matter density contrast $\delta_m$ are related via the Poisson equation $\nabla^2\phi = 4\pi G a^2\bar{\rho}_m\delta_m$.

Show that the spherical multipole coefficients $a^{\psi}_{lm}$ of the CMB lensing potential $\psi(\hat{\mathbf{n}}) = \sum_{lm} a^{\psi}_{lm} Y_{lm}(\hat{\mathbf{n}})$ are given by:

$$a^{\psi}_{lm} = 12\pi\Omega_{m,0}H_0^2 i^{-l}\int_0^{\chi_e} d\chi \int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}\frac{\chi_e - \chi}{\chi\chi_e}\left(\frac{\delta_m(\mathbf{k}, \tau)}{a(\tau)}\right) j_l(k\chi)Y^*_{lm}(\hat{\mathbf{k}}) \qquad (3)$$

*[Hint: it may be helpful to use the Rayleigh plane wave expansion:* $e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi\sum_{lm} i^l j_l(kx)Y^*_{lm}(\hat{\mathbf{k}})Y_{lm}(\hat{\mathbf{x}})$*.]*

**(d)** Derive an expression for the angular power spectrum of the CMB lensing potential, $C^{\psi}_l$, in terms of the matter power spectrum $P(k)$ at an early, initial time $\tau_i$. In deriving this expression, you should absorb the time evolution of $\delta_m(\mathbf{k}, \tau)$ into a linear growth factor $D_i(\tau)$, defined via $\delta_m(\mathbf{k}, \tau) = D_i(\tau)\delta_m(\mathbf{k}, \tau_i)$.

*[Hint: you may use the definition of the angular power spectrum* $\langle a_{lm}a^*_{l'm'}\rangle = C_l\delta_{l'l}\delta_{mm'}$*, the orthogonality relation* $\int d\hat{\mathbf{k}}Y^*_{lm}(\hat{\mathbf{k}})Y_{l'm'}(\hat{\mathbf{k}}) = \delta_{ll'}\delta_{mm'}$*, and the fact that the modes of the matter density contrast satisfy* $\delta^*_m(\mathbf{k}) = \delta_m(-\mathbf{k})$*.]*

**4** Consider a standard single-field slow-roll inflation model, where $\phi$ is the inflation field and $V(\phi)$ is its potential. You may assume throughout the entire problem that $a(\tau) = -(H\tau)^{-1}$ (with $\tau$ the conformal time) and that $H = \sqrt{\frac{V(\phi)}{3M_{\rm pl}^2}} \approx$ constant.

**(a)** Canonical quantization leads to the following expression for the field operator $\hat{f} = a\hat{\delta\phi}$, describing perturbations to the inflation field $\delta\phi$:

$$\hat{f}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[ f_{\mathbf{k}}^*(\tau) \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} + f_{\mathbf{k}}(\tau) \hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \right]$$

where $f_{\mathbf{k}}(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}(1 - \frac{i}{k\tau})$ and $\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}}^\dagger$ are lowering and raising operators. State the commutation relations obeyed by $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}'}^\dagger$. By calculating the two point correlation function of $\delta\phi$, deduce the dimensionless power spectrum of $\delta\phi$. Evaluate this spectrum when $k \ll aH$, and show that it is given by

$$\Delta_{\delta\phi}^2 = \left(\frac{H}{2\pi}\right)^2. \tag{1}$$

[*Hint: you may assume that the dimensionless power spectrum* $\Delta_{\delta\phi}^2$ *is related to the two point correlation function via* $\langle 0|\hat{\delta\phi}(\tau,\mathbf{x})\hat{\delta\phi}(\tau,\mathbf{x}+\mathbf{r})|0\rangle = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{2\pi^2}{k^3} \Delta_{\delta\phi}^2 e^{-i\mathbf{k}\cdot\mathbf{r}}$]

**(b)** The dimensionless power spectrum of the comoving curvature perturbation $\mathcal{R}$ in this inflation model is hence given by

$$\Delta_{\mathcal{R}}^2(k) = \frac{1}{2\epsilon M_{\rm pl}^2}\left(\frac{H}{2\pi}\right)^2, \tag{2}$$

where $\epsilon \equiv -\frac{d\ln H}{dN} = -\frac{\dot{H}}{H^2}$ is the first Hubble slow-roll parameter. Specify when the right hand side of this equation is to be evaluated; then show that the scalar spectral index $n_s \equiv 1 + \frac{d\ln\Delta_{\mathcal{R}}^2}{d\ln k}$ is given by

$$n_s - 1 = -2\epsilon - \eta, \tag{3}$$

where $\eta = \frac{d\ln\epsilon}{dN}$ is the second Hubble slow roll parameter.

**(c)** Consider a class of inflation models described by the potential

$$V(\phi) = \lambda M_{\rm pl}^4 \left(\frac{\phi}{M_{\rm pl}}\right)^\alpha, \tag{4}$$

with $\alpha$, $\lambda$ positive constants. For what $\phi$ values can these models support slow-roll inflation?

[*Hint: you may assume that for the Hubble slow-roll parameters* $\epsilon, \eta$ *the following holds during slow-roll inflation:* $\frac{M_{\rm pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 = \epsilon$ *and* $M_{\rm pl}^2\left(\frac{V_{,\phi\phi}}{V}\right) = 2\epsilon - \frac{\eta}{2}$*.*]

**(d)** The parameter $r$, which gives the ratio of power spectra of tensor perturbations and scalar perturbations, is related to the Hubble slow-roll parameter by $r = 16\epsilon$. For the class of models in (c), derive a relation between $n_s - 1$ and $r$. Given this relation, which ranges of observed $(r, n_s - 1)$ values are consistent with this class of models?

**END OF PAPER**